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Subdomain Finite Element Method with Quartic B-Splines for the Modified Equal Width Wave Equation[1]

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Abstract—In this paper, a numerical solution of the modified equal width wave (MEW) equation, has been obtained by a numerical technique based on Subdomain finite element method with quartic B-splines. Test problems including the motion of a single solitary wave and interaction of two solitary waves are studied to validate the suggested method. Accuracy and efficiency of the proposed method are discussed by computing the numerical conserved laws and error norms L_2 and L_∞. A linear stability analysis based on a Fourier method shows that the numerical scheme is unconditionally stable.

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1. INTRODUCTION

In this study, we will consider the following one dimensional modified equal width wave (MEW) equation

$$U_t + 3U^2U_x - \mu U_{xxt} = 0, \tag{1}$$

with the boundary conditions

$$\begin{aligned} U(a,t) &= 0, \quad U(b,t) = 0, \\ U_x(a,t) &= 0, \quad U_x(b,t) = 0, \\ U_{xx}(a,t) &= 0, \quad U_{xx}(b,t) = 0, \quad t > 0, \end{aligned} \tag{2}$$

and the initial condition

$$U(x,0) = f(x), \quad a \le x \le b,$$

where t is time, x is the space coordinate, μ is a positive parameter, $U(x, t)$ is wave amplitude and $f(x)$ is a prescribed function. The equal width wave (EW) equation, which is an alternative description of the nonlinear dispersive waves to the more usual Korteweg-de Vries (KdV) equation is a model nonlinear partial differential equation used for the simulation of one-dimensional non-linear waves propagating in dispersive media. Benjamin, Bona and Mahoney in [1] proposed the regularized long wave (RLW) equation to be a model for the same physical phenomena equally well as the KdV equation. MEW equation, which we discuss here, is related with the modified regularized long wave (MRLW) equation(see [2]) and modified Korteweg-de Vries (MKdV) equation (see [3]) and is based upon the equal width wave (EW) equation. This equation has solitary wave solutions with both positive and negative amplitudes, all of which have the same width. The MEW equation is a non-linear wave equation with cubic nonlinearity with a pulse-like solitary wave solution (see [4]). Analytical solutions of the MEW equation are known with only a restricted set of boundary and initial conditions. Therefore, many numerical methods have been used for solving the MEW equation with various boundary and initial conditions. Wazwaz in [5] investigated the MEW equation and two of its variants by the tanh and the sine-cosine methods. Zaki (see [6, 7]) considered the solitary wave interactions for the MEW equation by Petrov–Galerkin method using quintic B-spline finite elements and obtained the numerical solution of the EW equation by using the least-squares method. Vari-

[1] The article is published in the original.

ational iteration method is introduced to solve the MEW equation by J. Lu (see [8]). S.T. Mohyud-Din et al. solved the MEW equation numerically with homotopy perturbation method in [9[. Esen in [10, 11] applied a lumped Galerkin method based on quadratic B-spline finite elements have been used for solving the EW and MEW equation. A. Esen and S. Kutluay in [12] studied a linearized implicit finite difference method in solving the MEW equation. Saka in [13] proposed algorithms for the numerical solution of the MEW equation using quintic B-spline collocation method. T. Geyikli and S.B.G. Karakoç (see [14, 15]) solved the MEW equation by a collocation method using septic B-spline finite elements and using a Petrov–Galerkin finite element method with weight functions quadratic and element shape functions are cubic B-splines, Also S.B.G. Karakoc, solved the equation numerically with finite element methods (see 16]). D.J. Evans and K.R. Raslan in [17] studied the generalized EW equation by using collocation method based on quadratic B-splines to obtain the numerical solutions of a single solitary waves, and the birth of solitons. In this paper, we solve the MEW equation numerically by Subdomain finite element method using quartic B-splines. The performance and accuracy of the proposed method have been tested on two problems: the motion of a single solitary wave and the interaction of two solitary waves. A linear stability analysis based on a Fourier method shows that the numerical scheme is unconditionally stable.

2. QUARTIC B-SPLINE SOLUTION

For the numerical treatment, the solution domain of the problem is restricted over an interval $a \le x \le b$. Physical boundary conditions require $U \longrightarrow 0$ and $U_x \longrightarrow 0$ that for $x \longrightarrow \pm\infty$. The finite interval $[a, b]$ is partitioned into N finite elements of equal length h by the nodes x_m such that $a = x_0 < x_1 \ldots <= b$ and $h = (x_{m+1} - x_m)$. The quartic B-splines $\phi_m(x)$, $(m = -2(1)N + 1)$, at the knots x_m which form a basis over the interval $[a, b]$ are denned by the relationships (see [18])

$$\phi_m(x) = \frac{1}{h^4}\begin{cases} (x-x_{m-2})^4, & x \in [x_{m-2}, x_{m-1}], \\ (x-x_{m-2})^4 - 5(x-x_{m-1})^4, & x \in [x_{m-1}, x_m], \\ (x-x_{m-2})^4 - 5(x-x_{m-1})^4 + 10(x-x_m)^4, & x \in [x_m, x_{m+1}], \\ (x_{m+3}-x)^4 - 5(x_{m+2}-x)^4, & x \in [x_{m+1}, x_{m+2}], \\ (x_{m+3}-x)^4, & x \in [x_{m+2}, x_{m+3}], \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

A global approximation $U_N(x, t)$ to the exact solution $U(x, t)$ can be expressed in terms of the quartic B-splines as:

$$U_N(x, t) = \sum_{j=-2}^{N+1} \delta_j(t)\phi_j(x), \tag{4}$$

where δ_j are time dependent quantities to be determined from both boundary and weighted residual conditions. Each quartic B-spline covers five elements so that each element $[x_m, x_{m+1}]$ is covered by five splines. The nodal values U_m, U'_m, U''_m and U'''_m at the knots x_m are derived from (3) and (4) in the following form

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-2} + 11\delta_{m-1} + 11\delta_m + \delta_{m+1}, \\ U'_m &= U'(x_m) = \frac{4}{h}(-\delta_{m-2} - 3\delta_{m-1} + 3\delta_m + \delta_{m+1}), \\ U''_m &= U''(x_m) = \frac{12}{h^2}(\delta_{m-2} - \delta_{m-1} - \delta_m + \delta_{m+1}), \\ U'''_m &= U'''(x_m) = \frac{24}{h^3}(-\delta_{m-2} + 3\delta_{m+1} - 3\delta_m + \delta_{m+1}). \end{aligned} \tag{5}$$

A typical finite interval $[x_m, x_{m+1}]$ is mapped to the interval [0, 1] by local coordinates ξ related to the global coordinates

$$h\xi = x - x_m, \quad 0 \le \xi \le 1, \tag{6}$$

so the quartic B-spline shape functions over the element [0, 1] can be defined as

$$\phi^e = (\phi_{m-2}, \phi_{m-1}, \phi_m, \phi_{m+1}, \phi_{m+2}) = \begin{cases} \phi_{m-2} = 1 - 4\xi + 6\xi^2 - 4\xi^3 + \xi^4, \\ \phi_{m-1} = 11 - 12\xi - 6\xi^2 + 12\xi^3 - 4\xi^4, \\ \phi_m = 11 + 12\xi - 6\xi^2 - 12\xi^3 + 6\xi^4, \\ \delta_{m+1} = 1 + 4\xi + 6\xi^2 + 4\xi^3 - 4\xi^3, \\ \phi_{m+2} = \xi^4. \end{cases} \tag{7}$$

Since all splines apart from $\phi_{m-2}(x)$, $\phi_{m-1}(x)$, $\phi_m(x)$, $\phi_{m+1}(x)$ and $\phi_{m+2}(x)$ are zero over the element [0, 1], approximation (4) over this element can be written in terms of basis functions (7) as

$$U_N(\xi, t) = \sum_{j=m-2}^{m+2} \delta_j(t)\phi_j(\xi),$$

where $\delta_{m-2}, \delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2}$ act as element parameters and B-splines $\phi_{m-2}(x), \phi_{m-1}, \phi_m, \phi_{m+1}, \phi_{m+2}$ as element shape functions. When Subdomain method is applied to Eq. (1) with weight function

$$W_m(x) = \begin{cases} 1, & x \in [x_m, x_{m+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

we obtain the weak form of (1)

$$\int_{x_m}^{x_{m+1}} 1.[U_t + 3U^2U_x - \mu U_{xxt} dx] = 0. \tag{9}$$

Substituting the transformation (6) into weak form (9) and integrating Eq. (9) term by term with some manupulation by parts, leads to

$$\begin{aligned} &\frac{h}{5}(\dot{\delta}_{m-2} + 26\dot{\delta}_{m-1} + 66\dot{\delta}_m + 26\dot{\delta}_{m+1} + \dot{\delta}_{m+2}) + Z_m(-\delta_{m-2} - 10\delta_{m-1} \\ &+ 10\delta_{m+1} + \delta_{m+2}) - \frac{4\mu}{h}(\dot{\delta}_{m-2} + 2\dot{\delta}_{m-1} - 6\dot{\delta}_m + 2\dot{\delta}_{m+1} + \dot{\delta}_{m+2}) = 0, \end{aligned} \tag{10}$$

where the dot denotes differentiation with respect to t and

$$Z_m = 3(\delta_{m-2} + 11\delta_{m-1} + 11\delta_m + \delta_{m+1})^2.$$

If time parameters δ_m and its time derivatives $\dot{\delta}_m$ in Eq. (10) are discretized by the Crank–Nicolson and forward difference approach respectively,

$$\delta_m = \frac{\delta_m^n + \delta_m^{n+1}}{2}, \quad \dot{\delta}_m = \frac{\delta_m^{n+1} - \delta_m^n}{\Delta t} \tag{11}$$

we obtain a recurrence relationship between two time levels n and $n + 1$ relating two unknown parameters δ_i^{n+1}, δ_i^n, $i = m - 2, m - 1, \ldots, m + 2$,

$$\begin{aligned} &\alpha_{m1}\delta_{m-2}^{n+1} + \alpha_{m2}\delta_{m-1}^{n+1} + \alpha_{m3}\delta_m^{n+1} + \alpha_{m4}\delta_{m+1}^{n+1} + \alpha_{m5}\delta_{m+2}^{n+1} \\ &= \alpha_{m5}\delta_{m-2}^n + \alpha_{m4}\delta_{m-1}^n + \alpha_{m3}\delta_m^n + \alpha_{m2}\delta_{m+1}^n + \alpha_{m1}\delta_{m+2}^n, \quad m = 0, 1, \ldots, N, \end{aligned} \tag{12}$$

where

$$\alpha_{m1} = 1 - EZ_n - M, \quad \alpha_{m2} = 26 - 10EZ_m - 2M, \quad \alpha_{m3} = 66 + 6M,$$
$$\alpha_{m4} = 26 + 10EZ_m - 2M, \quad \alpha_{m5} = 1 + EZ_m - M,$$

and

$$E = \frac{5\Delta t}{2h}, \quad M = \frac{20\mu}{h^2}.$$

The system (12) consists of $N + 1$ linear equation in $N + 5$ unknowns (δ_{-2}, δ_{-1}, ..., δ_{N+1}, δ_{N+2}). To get a solution to this system, we need four additional constraints. These are obtained from the boundary conditions (2) and can be used to eliminate δ_{-1}, δ_{-1}, δ_{N+1} and δ_{N+2} from the system (12) which then becomes a matrix equation for the δ_{N+1} unknowns $d = (\delta_0, \delta_1, ..., \delta_N)$ of the form

$$Ad^{n+1} = Bd^n.$$

A lumped value for Z_m is obtained from $(U_m + U_{m+1})^2/4$ as

$$Z_m = \frac{3}{4}(\delta_{m-2} + 12\delta_{m-1} + 22\delta_m + 12\delta_{m+1} + \delta_{m+2})^2.$$

The resulting system can be efficiently solved with a variant of the Thomas algorithm, and we need an inner iteration $\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ at each time step to cope with the non-linear term Z_m. A typical member of the matrix system (12) can be written in terms of the nodal parameters δ_m^n as

$$\begin{aligned}&\gamma_1\delta_{m-2}^{n+1} + \gamma_2\delta_{m-1}^{n+1} + \gamma_3\delta_m^{n+1} + \gamma_4\delta_{m+1}^{n+1} + \gamma_5\delta_{m+2}^{n+1}\\ &= \gamma_5\delta_{m-2}^n + \gamma_4\delta_{m-1}^{n+} + \gamma_3\delta_m^n + \gamma_2\delta_{m+1}^n + \gamma_1\delta_{m+2}^n,\end{aligned} \tag{13}$$

where

$$\gamma_1 = \alpha - \beta - \lambda, \quad \gamma_2 = 26\alpha - 10\beta - 2\lambda, \quad \gamma_3 = 66\alpha + 6\lambda,$$
$$\gamma_4 = 26\alpha + 10\beta - 2\lambda, \quad \gamma_5 = \alpha + \beta - \lambda$$

and

$$\alpha = 1, \quad \beta = EZ_m, \quad \lambda = M.$$

Before the solution process begin iteratively, the initial vector $\delta^0 = (\delta_0, \delta_1, \ldots, \delta_N)$ must be determined by using the initial condition and following derivatives at the boundaries:

$$U'(a, 0) = \frac{4}{h}(-\delta_{-2}^0 - 3\delta_{-1}^0 + 3\delta_0^0 + \delta_1^0) = 0,$$

$$U''(a, 0) = \frac{12}{h^2}(\delta_{-2}^0 - \delta_{-1}^0 - \delta_0^0 + \delta_1^0) = 0,$$

$$U(x_m, 0) = \delta_{m-2}^0 + 11\delta_{m-1}^0 + 11\delta_m^0 + \delta_{m+1}^0 = f(x), \quad m = 0, 1, \ldots, N-1$$

$$U'(b, 0) = (-\delta_{N-2}^0 - 3\delta_{N-1}^0 + 3\delta_N^0 + \delta_{N+1}^0) = 0,$$

$$U''(a, 0) = (\delta_{N-2}^0 - \delta_{N-1}^0 - \delta_N^0 + \delta_{N+1}^0) = 0.$$

Eliminating δ_{-2}^0 δ_{-1}^0, δ_{N+1}^0 from the system (12) we get $(N + 1) \times (N + 1)$ four-banded matrix system of the form:

$$W\delta^0 = K,$$

where W is

$$W = \begin{bmatrix} 18 & 6 & & & & & & \\ 11.5 & 11.5 & 1 & & & & & \\ & 1 & 11 & 11 & 1 & & & \\ & & & & & & & \\ & & & & 1 & 11 & 11 & 1 \\ & & & & & 2 & 14 & 8 \end{bmatrix},$$

$\delta^0 = [\delta_0^0, \delta_1^0, \ldots, \delta_N^0]^{\mathrm{T}}$ and $K = [U(x_0, 0), U(x_1, 0), U(x_N, 0)]^{\mathrm{T}}$. This matrix system can be solved efficiently by using a variant of Thomas algorithm.

2.1. A Linear Stability Analysis

We implement the Fourier stability analysis method in which the growth factor of a typical fourier mode is defined as

$$\delta_j^n = \xi^n e^{ijkh} \tag{14}$$

where k is mode number and h the element size. To apply the stability analysis, the MEW equation needs to be linearized by assuming that the quantity U in the non-linear term U^2U_x is locally constant. Substituting the Eq. (14) into the scheme (13) we have

$$g = \frac{a - ib}{a + ib}, \tag{15}$$

where

$$\begin{aligned} a &= 33 + 3\lambda + (26 - 2\lambda)\cos kh + (1 - \lambda)\cos 2kh, \\ b &= 10\beta\sin(kh) + \beta\sin(2kh). \end{aligned} \tag{16}$$

Taking the modulus of Eq. (15) gives $|g| = 1$, therefore we find that the scheme (13) is unconditionally stable.

3. NUMERICAL EXAMPLES AND RESULTS

Now we obtain the numerical solutions of the modified equal width wave equation for two problems: the motion of single solitary wave and interaction of two solitary waves. All numerical calculations were performed in double precision arithmetic on a Pentium 4 PC using a Fortran compiler. The accuracy of the method is measured by both the L_2 error norm

$$L_2 = \left\|U^{\text{exact}} - U_N\right\|_2 \simeq \sqrt{h\sum_{J=1}^{N}\left|U_j^{\text{exact}} - (U_N)_j\right|^2},$$

and the error norm

$$L_\infty = \left\|U^{\text{exact}} - U_N\right\|_\infty \simeq \max_j\left|U_j^{\text{exact}} - (U_N)_j\right|,$$

to show how good the numerical scheme predicts the position and amplitude of the solution as the simulation proceeds. The MEW Eq. (1) satisfies only three conservation laws given by [19]

$$I_1 = \int_0^b U dx \simeq h\sum_{J=1}^{N} U_j^n,$$

$$I_2 = \int_a^b [U^2 + \mu(U_x)^2]dx \simeq h\sum_{J=1}^{N}[(U_j^n)^2 + \mu(U_x)_j^n],$$

Table 1. Invariants and error norms for single solitary wave with $h = 0.1$, $\Delta t = 0.05$, $A = 0.25$

t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	0.7853966	0.1666664	0.0052083	0.0000000	0.0000000
5	0.7853966	0.1666664	0.0052083	0.0129509	0.0077917
10	0.7853966	0.1666664	0.0052083	0.0259120	0.0157724
15	0.7853967	0.1666664	0.0052083	0.0388849	0.0239295
20	0.7853967	0.1666664	0.0052083	0.0518731	0.0321136
20 [11]	0.7853898	0.1667614	0.0052082	0.0796940	0.0465523
20 [12]	0.7853977	0.1664735	0.0052083	0.2692812	0.2569972
20 [15]	0.7853967	0.1666663	0.0052083	0.0801465	0.0461218
20 [17]	0.7849545	0.1664765	0.0051995	0.2905166	0.2498925

$$I_3 = \int_a^b U^4 dx \simeq h \sum_{J=1}^{N} (U_j^n)^4,$$

which correspond to conservation of mass, momentum and energy, respectively. Olver in [20] has shown that EW equation have just three such laws and this may also be true of the MEW equation also.

3.1. The Motion of Single Solitary Wave

As a first problem we consider Eq. (1) with the boundary conditions $U \longrightarrow 0$ as $x \longrightarrow \pm\infty$ and the initial condition

$$U(x, 0) = A\operatorname{sec}h(k[x - x_0]).$$

The MEW equation has an analytical solution of the form

$$U(x, t) = A\operatorname{sec}h(k[x - x_0 - vt]),$$

where $k = \sqrt{\frac{1}{\mu}}$ and $v = \frac{A^2}{2}$ is the wave velocity. This solution corresponds to a solitary wave of amplitude A, initially centered at x_0. For this problem the analytical values of the invariants are (see [6])

$$I_1 = \frac{A\pi}{k}, \quad I_2 = \frac{2A^2}{k} + \frac{2\mu k A^2}{3}, \quad I_3 = \frac{4A^4}{3k}. \tag{17}$$

For the computational work we have used the parameters $h = 0.1$, $\Delta t = 0.05$, $\mu = 1$, $x_0 = 30$, $A = 0.25$ through the interval $0 \le x \le 80$. The analytical values of invariants are $I_1 = 0.7853982$, $I_2 = 0.1666667$, $I_3 = 0.0052083$. The simulations are run to time $t = 20$ to find error norms L_2, L_∞ and conserved quantities I_1, I_2, I_3. Table 1 compares the values of the invariants and error norms obtained by the present method and

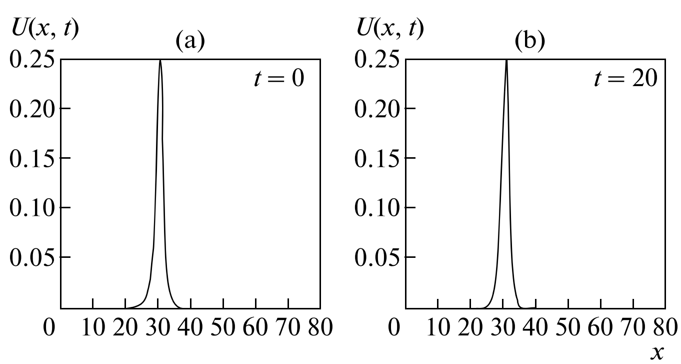

Fig. 1. The motion of a single solitary wave with $h = 0.1$, $t = 0.05$ at $t = 0$ and $t = 20$.

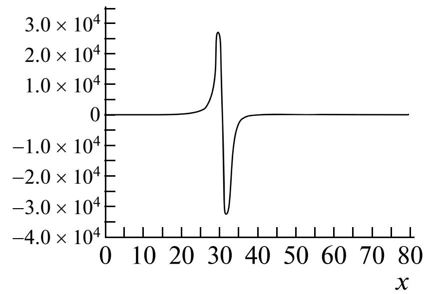

Fig. 2. Error graph at $t = 20$.

some earlier methods (see [11, 12, 15, 17]) at time $t = 20$. It is clear from the table that the error norms obtained by the present method are smaller than those given in [11, 12, 15, 17] and agreement between analytical and numerical solutions is perfect. Invariants I_1, I_2 and I_3 change by less than $2 \times 10^{-6}\%$, $3 \times 10^{-6}\%$, $7 \times 10^{-6}\%$ throught the run, respectively. The invariants thus remain satisfactorily constant. Figure 1 shows that the proposed method perform the motion of propagation of a solitary wave satisfactorily, which moves to the right at a constant speed and preserves its amplitude and shape with increasing time as expected. Amplitude is 0.25 at $t = 0$ which is located at $x = 30$, while it is 0.249922 at $t = 20$ which is located at $x = 30.6$. The absolute difference in amplitudes at times $t = 0$ and $t = 20$ is 7.8×10^{-5} so that there is a little change between amplitudes. The error graph is given at $t = 20$ in Fig. 2. As is seen that the maximum errors occur around the central position of the solitary wave. This problem is also considered for different values

Table 2. Invariants and error norms for single solitary wave with different amplitudes, $h = 0.1$, $t = 0.01$

A	t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.25	0	0.7853966	0.1666664	0.0052083	0.0000000	0.0000000
	5	0.7853966	0.1666664	0.0052083	0.0129353	0.0077822
	10	0.7853966	0.1666664	0.0052083	0.0258808	0.0157541
	15	0.7853966	0.1666664	0.0052083	0.0388380	0.0239013
	20	0.7853967	0.1666664	0.0052083	0.0518107	0.0320756
	20 [12]	–	–	–	0.2692249	0.2569562
0.50	0	1.5707932	0.6666656	0.0833330	0.0000000	0.0000000
	5	1.5707931	0.6666656	0.0833330	0.1036996	0.0641990
	10	1.5707931	0.6666656	0.0833330	0.2081535	0.1320066
	15	1.5707930	0.6666655	0.0833330	0.3140373	0.2014618
	20	1.5707930	0.6666655	0.0833330	0.4215203	0.2711694
	20 [12]	–	–	–	1.8266059	1.4575680
0.75	0	2.3561897	1.4999976	0.4218734	0.0000000	0.0000000
	5	2.3561896	1.4999974	0.4218733	0.3528836	0.2247579
	10	2.3561894	1.4999973	0.4218732	0.7164949	0.4603957
	15	2.3561894	1.4999973	0.4218732	1.0954564	0.6982392
	20	2.3561893	1.4999973	0.4218732	1.4814601	0.9362386
	20 [12]	–	–	–	4.3957110	3.0917930
1.0	0	3.1415863	2.6666625	1.3333283	0.0000000	0.0000000
	5	3.1415855	2.6666614	1.3333272	0.8533818	0.5485847
	10	3.1415849	2.6666607	1.3333265	1.7614062	1.1176331
	15	3.1415844	2.6666600	1.3333258	2.6889848	1.6874848
	20	3.1415838	2.6666592	1.3333251	3.6194880	2.2576829
	20 [12]	–	–	–	8.2853140	5.6821310

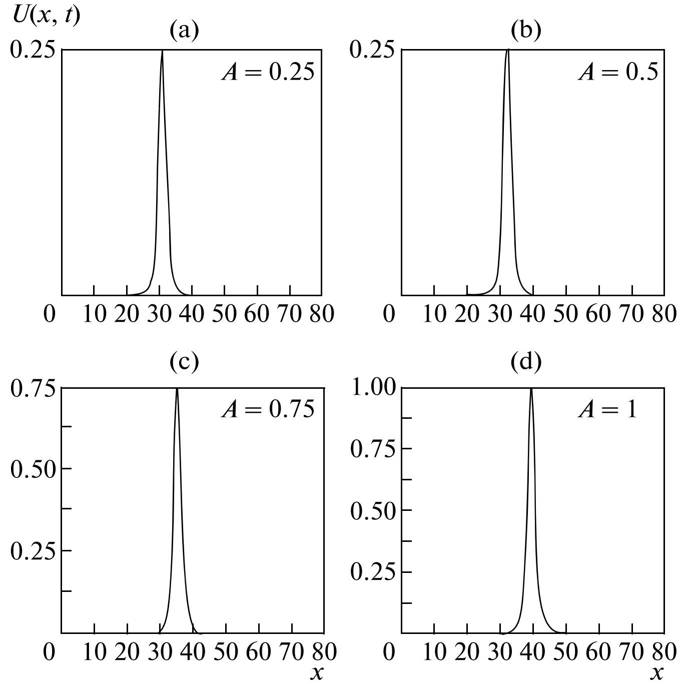

Fig. 3. Single solitary wave solutions for various values of A at $t = 20$.

of the amplitude at $h = 0.1$ and $t = 0.01$. In Table 2 the error norms and invariants are listed for $A = 0.25$, 0.5, 0.75, 1. A comparison with [12] shows that the present method provide better results in terms of the error norms L_2 and L_∞. Figure 3 shows the solutions of the single solitary wave with $h = 0.1$, $\Delta t = 0.01$ for different values of amplitude A at time $t = 20$. It is clear that the soliton moves to the right at a constant speed and almost preserve its amplitude and shape with an increasing of time, as expected. We compute the convergence rates for the numerical method in space steps h_m and time steps Δt_m, with the following formulas respectively (see [11]):

$$\text{order} = \frac{\log_{10}\left(\left|U^{\text{exact}} - U^{\text{num}}_{h_m}\right| / \left|U^{\text{exact}} - U^{\text{num}}_{h_{m+1}}\right|\right)}{\log_{10}(h_m/h_{m+1})}$$

and

$$\text{order} = \frac{\log_{10}\left(\left|U^{\text{exact}} - U^{\text{num}}_{\Delta t_m}\right| / \left|U^{\text{exact}} - U^{\text{num}}_{\Delta t_{m+1}}\right|\right)}{\log_{10}(\Delta t_m/\Delta t_{m+1})}.$$

The time rate of convergence for the numerical method is computed with various time step and fixed space step are given in Table 3. It is concluded from the table that the present method provides remarkable reductions in convergence rates for the smaller times. In addition, the space rate of the convergence for

Table 3. The order of convergence at $t = 20$, $h = 0.1$, $A = 0.25$, $0 \le x \le 80$

Δt_m	$L_2 \times 10^3$	Order	$L_\infty \times 10^3$	Order
0.8	0.06867606	–	0.04220891	–
0.4	0.05598789	0.29469246	0.03460578	0.28653455
0.2	0.0528502	0.08320806	0.03270686	0.08141976
0.1	0.05206843	0.02149784	0.03223226	0.02108791
0.05	0.05187319	0.00541981	0.03211361	0.00532049
0.025	0.05182440	0.00135758	0.03208395	0.00133308

Table 4. The order of convergence at $t = 20$, $\Delta t = 0.05$, $A = 0.25$, $0 \le x \le 80$

h_m	$L_2 \times 10^3$	Order	$L_\infty \times 10^3$	Order
0.8	4.71902836	–	4.41397880	–
0.4	0.83089313	2.50575500	0.49959156	3.14325869
0.2	0.20634992	2.00957005	0.12731249	1.97237514
0.1	0.05187319	1.99203188	0.03211361	1.98711720
0.05	0.01324799	1.96921560	0.00810628	1.98607293
0.025	0.00325160	2.02655179	0.00204980	1.98355685

Table 5. Comparision of invariants for the interaction of two solitary waves with results from [11] with $h = 0.1$, $t = 0.025$

t	Present method			[11]		
	I_1	I_2	I_3	I_1	I_2	I_3
0	4.7123733	3.3333305	1.4166643	4.7123884	3.3352890	1.4166697
10	4.7123741	3.3333303	1.4166642	4.7123853	3.3352836	1.4166647
20	4.7123835	3.3333399	1.4166762	4.7123748	3.3353041	1.4166926
30	4.7126442	3.3336245	1.4170020	4.7126410	3.3359464	1.4176398
40	4.7123933	3.3333936	1.4168064	4.7123946	3.3355951	1.4170695
50	4.7121712	3.3332154	1.4166420	4.7121567	3.3351175	1.4165797
55	4.7121725	3.3332189	1.4166412	4.7121400	3.3350847	1.4165527
60	4.7122144	3.3332433	1.4166475	–	–	–
70	4.7123166	3.3332928	1.4166608	–	–	–
80	4.7123744	3.3333167	1.4166672	–	–	–

the numerical method is computed with various space step and fixed time step are recorded in Table 4. We have clearly seen that the convergence rates when $\Delta t = 0.05$ is fixed are not as good as for the space steps.

3.2. Interaction of Two Solitary Waves

Our second study concerns with the interaction between two solitary waves. To study interaction of two solitary waves we use the boundary conditions $U \longrightarrow 0$ as $x \longrightarrow \pm\infty$ and the initial condition

$$U(x, 0) = \sum_{j=1}^{2} A_j \mathrm{sec}h(k[x - x_j]), \tag{18}$$

where $k = \sqrt{\frac{1}{\mu}}$. The initial condition (18) represents two solitary waves, one with amplitude A_1 placed initially at $x = x_1$ and the second with amplitude A_2 placed at $x = x_2$. For the numerical simulation, we first choose the parameters $h = 0.1$, $\mu = 1$, $\Delta t = 0.025$, $A_1 = 1$, $A_2 = 0.5$, $x_1 = 15$, $x_2 = 30$ over the interval $0 < x < 80$ to coincide with those used by Esen in [11]. The analytical invariants can be found as

$$I_1 = \pi(A_1 + A_2) = 4.7123889, \quad I_2 = \frac{8}{9}(A_1^2 + A_2^2) = 3.3333333, \\ I_3 = \frac{4}{3}(A_1^4 + A_2^4) = 1.4166667. \tag{19}$$

The calculations are performed from $t = 0$ to $t = 80$ and values of the invariant quantities I_1, I_2 and I_3 are recorded in Table 5. Table 5 displays a comparision of the values of the invariants obtained by the present

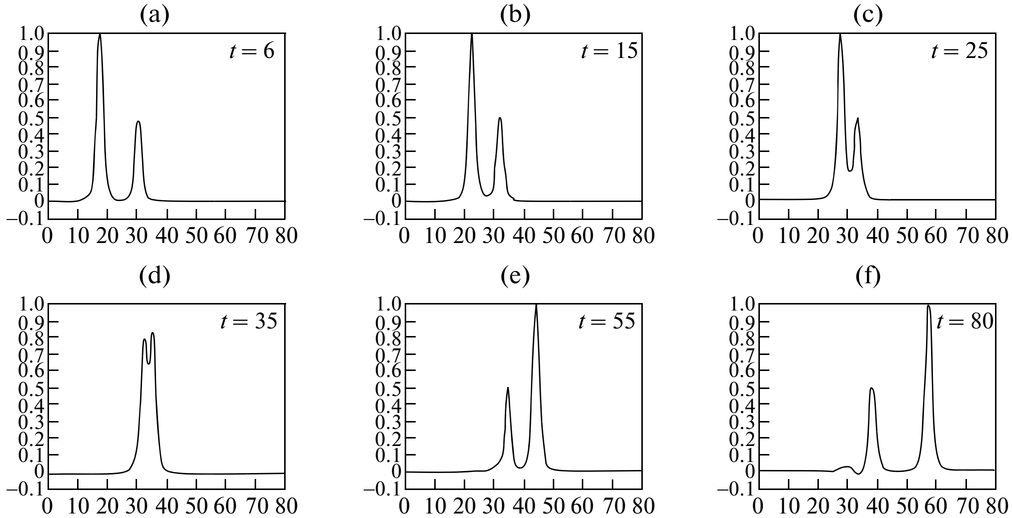

Fig. 4. Interaction of two solitary waves with $A_1 = 1$, $A_2 = 0.5$, $0 \le x \le 80$.

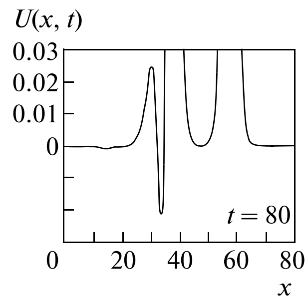

Fig. 5. An expanded vertical scale of Fig. 4f.

method with those obtained in [11]. It is seen that the obtained values of the invariants remain almost constant during the computer run. They are found to be close to values given by (19). The absolute difference between the values of the invariants obtained by the present method at times $t = 0$ and $t = 80$ are $\Delta I_1 = 1.1 \times 10^{-6}$, $\Delta I_2 = 1.38 \times 10^{-5}$, $\Delta I_3 = 2.4 \times 10^{-6}$ whereas they are $\Delta I_1 = 2.48 \times 10^{-4}$, $\Delta I_2 = 2.04 \times 10^{-4}$, $\Delta I_3 = 1.17 \times 10^{-4}$ in [11]. The changes in these quantities I_1, I_2, I_3 are less than $2.4 \times 10^{-5}\%$, $4.1 \times 10^{-4}\%$, $2 \times 10^{-4}\%$ during the run so that the numerical algorithm has good conservation properties. Figure 4 illustrates the behaviour of the interaction of two positive solitary waves. It is observed from the Fig. 4 at $t = 0$ the wave with larger amplitude is on the left of the second wave with smaller amplitude. Since the taller wave moves faster than the shorter one, it catches up and collides with the shorter one at $t = 35$ and then moves away from the shorter one as time increases. When the interaction is completed we get the figure at $t = 80$. At $t = 80$, the amplitude of larger waves is 1.000331 at the point $x = 56.9$ whereas the amplitude of the smaller one is 0.498729 at the point $x = 37.7$. It is found that the absolute difference in amplitude is 0.331×10^{-3} for the smaller wave and 0.127×10^{-2} for the larger wave for this algorithm. At $t = 80$, we saw an oscillation of small amplitude trailing behind the solitary waves. To see this oscillation the scale of figure at $t = 80$ magnified as shown in Fig. 5. It is clearly seen from the Fig. 5 that an oscillation of the small amplitude is trailing behind the solitary waves.

In addition we have studied the interaction of two solitary waves with the parameters $\mu = 1$, $x_1 = 15$, $x_2 = 30$, $A_1 = -2$, $A_2 = 1$, $h = 0.1$ and $\Delta t = 0.025$ in the range $0 \le x \le 150$. Figure 6 shows the development of the solitary wave interaction. As is clearly seen from the Fig. 6, at $t = 0$ a wave with negative amplitude is on the left of another wave with positive amplitude. The larger wave with the negative amplitude catches up with the smaller one with the positive amplitude as the time increases. Table 6 displays a comparision

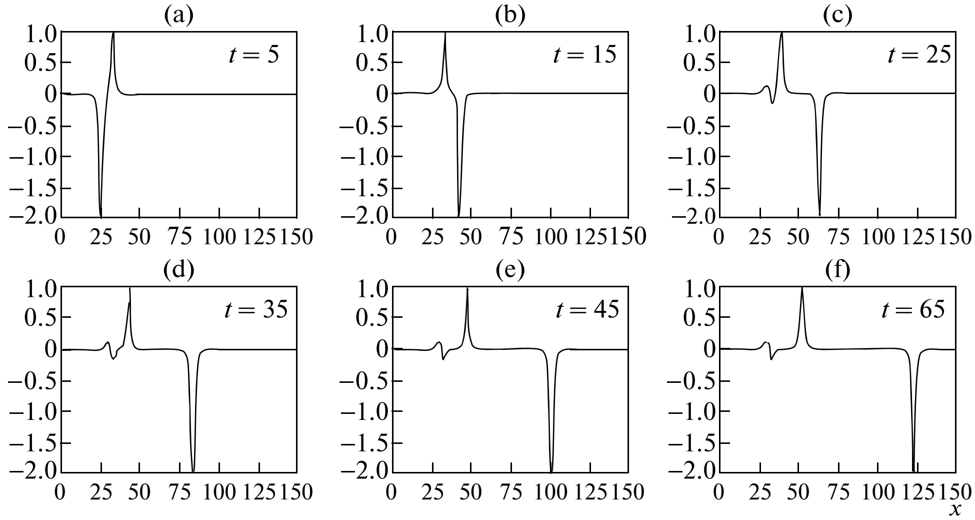

Fig. 6. Interaction of two solitary waves with $h = 0.1$, $A_1 = -2$, $A_2 = 1$.

of the values of the invariants obtained by the present method with those obtained in [11]. At $t = 55$, the amplitude of the smaller wave is 0.973753 at the point $x = 52.5$, whereas the amplitude of the larger one is −2.002144 at the point $x = 123.5$. It is found that the absolute difference in amplitudes is 2.62×10^{-2} for the smaller wave and 2.14×10^{-3} for the larger wave. It is observed that the obtained values of the invariants remain almost constant during the computer run. The absolute difference between the values of the invariants obtained by the present method at times $t = 0$ and $t = 55$ are $\Delta I_1 = 3.3 \times 10^{-4}$, $\Delta I_2 = 5.66 \times 10^{-4}$, $\Delta I_3 = 2.33 \times 10^{-3}$ whereas they are $\Delta I_1 = 5.1 \times 10^{-2}$, $\Delta I_2 = 1.38 \times 10^{-1}$, $\Delta I_3 = 5.59 \times 10^{-1}$ in [11]. The analytical invariants can be found by using Eq. (19) as

$$I_1 = -3.1415927, \quad I_2 = 13.3333333, \quad I_3 = 22.6666667.$$

The changes of invariants I_1, I_2, I_3 are less than $10 \times 10^{-3}\%$, $4 \times 10^{-3}\%$, $10 \times 10^{-3}\%$ during the run, respectively. The invariants so remain constant.

Table 6. Comparision of invariants for the interaction of two solitary waves with results from [11] with $h = 0.1$, $t = 0.025$ in the region $0 \le x \le 150$

t	Present method			[11]		
	I_1	I_2	I_3	I_1	I_2	I_3
0	−3.1415739	13.3333023	22.6665313	−3.1415915	13.3411364	22.6666177
5	−3.1415386	13.3333028	22.6665164	−3.1373341	13.3297086	22.6211074
10	−3.1294939	13.3157754	22.5691062	−3.1165140	13.2819575	22.3386157
15	−3.1429719	13.3321947	22.6660670	−3.1243642	13.2879992	22.4502917
20	−3.1418073	13.3334504	22.6675388	−3.1190016	13.2781110	22.4081976
25	−3.1416446	13.3335997	22.6678271	−3.1147243	13.2672538	22.3644947
30	−3.1416683	13.3336485	22.6680086	−3.1106562	13.2563740	22.3211768
35	−3.1417159	13.3336973	22.6681813	−3.1065564	13.2454531	22.2776978
40	−3.1417595	13.3337361	22.6683524	−3.1025255	13.2346133	22.2346619
45	−3.1418096	13.3337830	22.6685236	−3.0985577	13.2238575	22.1921206
50	−3.1418535	13.3338213	22.6686945	−3.0945539	13.2130371	22.1493051
55	−3.1419037	13.3338680	22.6688656	−3.0905294	13.2023061	22.1067310

4. CONCLUSIONS

In this paper, a numerical solution of the MEW equation with a variant of initial and boundary conditions was obtained using the Subdomain method based on the quartic B-splines. We tested our scheme through single solitary wave in which the analytic solution is known and extended it to study the interaction of two solitary waves where the analytical solution is unknown during the interaction. To show how good and accurate the numerical solutions of the test problems we have calculated the error norms L_2 and L_∞. The obtained results show that the Subdomain method involving quartic B-spline shape function is a remarkably successful numerical technique for solving the MEW equation and also can be efficiently applied to a broad class of physically important non-linear partial differential equations.

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