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Fractional-Order Analysis of Modified Chua's Circuit System with the Smooth Degree of 3 and Its Microcontroller-Based Implementation with Analog Circuit Design

Junxia Wang [1], Li Xiao [1], Karthikeyan Rajagopal [2], Akif Akgul [3,*], Serdar Cicek [4] and Burak Aricioglu [5]

[1] School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China; wangjunxia@cug.edu.cn (J.W.); xiaoli@cug.edu.cn (L.X.)

[2] Center for Nonlinear Systems, Chennai Institute of Technology, Chennai 600069, India; karthikeyan.rajagopal@citchennai.net

[3] Department of Computer Engineering, Faculty of Engineering, Hitit University, Corum 19030, Turkey

[4] Department of Electronic & Automation, Vocational School of Hacibektas, Nevsehir Haci Bektas Veli University, Nevsehir 50800, Turkey; serdarcicek@gmail.com or serdarcicek@nevsehir.edu.tr

[5] Department of Electrical and Electronics Engineering, Faculty of Technology, Sakarya University of Applied Sciences, Sakarya 54187, Turkey; baricioglu@sakarya.edu.tr

* Correspondence: akifakgul@hitit.edu.tr

Abstract: In the paper, we futher consider a fractional-order system from a modified Chua's circuit system with the smooth degree of 3 proposed by Fu et al. Bifurcation analysis, multistability and coexisting attractors in the the fractional-order modified Chua's circuit are studied. In addition, microcontroller-based circuit was implemented in real digital engineering applications by using the fractional-order Chua's circuit with the piecewise-smooth continuous system.

Keywords: modified Chua's circuit; stability; bifurcation; fractional-order analysis; microcontroller-based implementation

1. Introduction

In the first half of the last century, the analysis of oscillatory systems developed by nonlinear oscillation theory was emphasized slowly. Based on the numerical method, some classical attractors are found with unstable equilibria [1–4]. These attractors can be found from the neighborhood of equilibrium and be evolute of the local unstable manifold. Recently, Leonov et al. [5–7] proposed a 'hidden attractor' where there are no neighborhoods of equilibria in the basin of attraction. By investigating hidden oscillations, Leonov et al. discovered hidden Chua attractor [5], which can be used as the experimental vehicle for chaotic and nonlinear research. However, there exist some critical issues in Chua's circuit for chaotic application, such as control issues, circuit implementation, and fractional-order analysis. As shown in the Lorenz system and the Henon map, smooth quadratic functions are considered. It is therefore very reasonable to ask what the application is for piecewise quadratic functions in some chaotic models.

Nowadays, digital designs of chaotic systems and applications provide convenience for engineering applications. Microcontrollers are preferred in chaotic system-based applications because of their high performance and relatively inexpensive microcontrollers. As we know, microcontrollers can be used in chaos applications such as chaos-based encryption [8–10], random number generators [10,11], chaos-based communication [12], chaotic synchronization [13]. Chaotic systems were used in some applied fields [14,15]. If microcontroller-based realization of fractional-order chaotic systems is provided, it can be used easily in these applications. Therefore, the microcontroller-based implementation of the fractional-order 3D Chua's circuit system is also applied.

In this paper, improved circuit implementation of a 3D modified Chua's system is given. Analyses are carried out for the fractional-order from the modified system

to investigate the dynamical behaviors. Section 3 shows the form of fractional-order 3D Chua's circuit. Section 4 gives the results about bifurcation analysis, route to chaos, multistability and coexisting attractors in the fractional-order model. Section 5 implements the microcontroller-based circuit in real digital engineering applications. The 3D Chua's circuit with function $x|x|$ is implemented by simulation environment in Section 6. In the final section, the results and conclusion are provided.

2. The Modified Chua's Circuit System with the Smooth Degree of 3

By using the function $x|x|$ in the circuit, Prof. Chen et al. considered the following modified Chua's circuit system [16]:

$$
\begin{aligned}
\dot{x} &= \alpha(y - g(x)), \\
\dot{y} &= x - y + z, \\
g(x) &= ax + bx|x|, \\
\dot{z} &= -\beta y.
\end{aligned}
\tag{1}
$$

Here α, β, a and b represent the system (1)'s parameters. System (1) is piecewise smooth and first-order differentiable at the boundary on the switching interface. Moreover, the smooth degree of the system at the equilibrium $(0, 0, 0)$ is three [17]. Therefore, compared with smooth Chua's circuit, system (1) is needed to further be studied and some new results can occur due to the smoothness property [16–18]. If $\alpha = 9.267, \beta = 14, a = -1/6$, and $b = 1/16$, a chaotic attractor given in Figure 1 can be obtained [16].
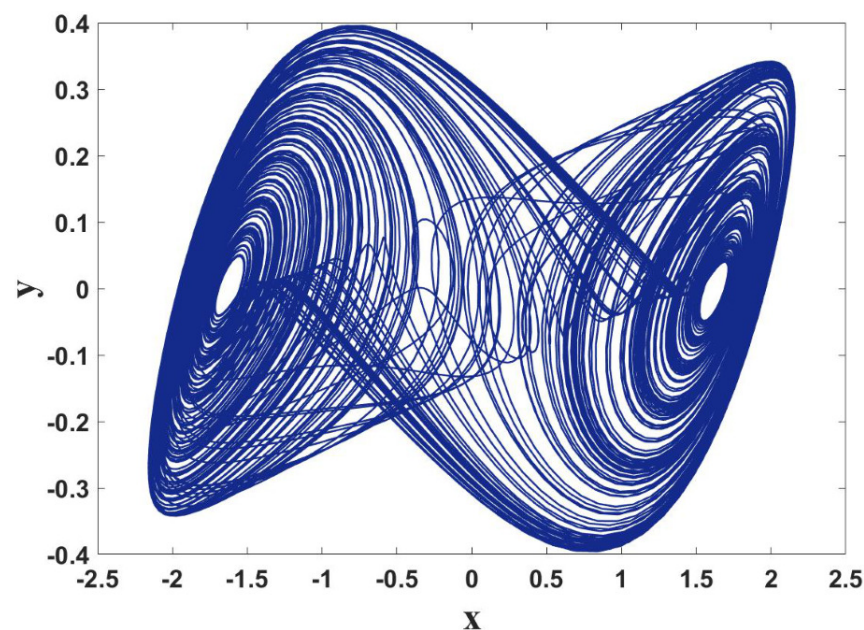
Figure 1. When $\alpha = 9.267, \beta = 14, a = -1/6, b = 1/16$, chaos can be obtained in system (1) with initial values $(-1.01, -0.01, -0.01)$.

3. Fractional-Order Model of Modified Chua's Circuit

The Caputo method is one of the most common methods used in Solving the fractional-order systems numerically. However, the Grunwald–Letnikov (GL) method can be preferred for numerical solutions of the fractional-order systems because of the smoothness of the resultant approximations [19,20]. Thus, we choose the GL method in this section because of its iterative feature. The memory effect can be observed in the sum and binomial coefficients are defined recursively and have smooth properties. Using the continuous Riemann–Liouville method for the discretization of the fractional equations [21–23], the GL derivative can be used and requires a discrete convolution of binomial coefficient function

and the function of interest. Here, the binomial coefficients are defined analytically [3]. Some related results can be found in refereces [24–30].

The definition of GL derivative is

$$
\begin{aligned}
*[^{GL}_{a}]D^q_t f(t) \ &= \lim_{h\to 0} \sum_{j=0}^{\frac{t-a}{h}} (-1)^n \binom{q}{j} f(t-jh) \\
&= \lim_{h\to 0} \frac{1}{h^q} \Delta^q_h f(t)
\end{aligned}
\tag{2}
$$

and

$$
*[^{GL}_{t-L}]D^q_t f(t) \ = \lim_{h\to 0} \frac{1}{h^q} \sum_{j=0}^{N(t)} b_j f(t-jh),
\tag{3}
$$

where a and t represent the limits, $\Delta^q_h f(t)$ represents generalized difference, h is the step size and q is the fractional-order. In order to limit the memory for binomial coefficients, short memory principle is used.

$$
N = \min\left\{ \left[\frac{t}{h}\right], \left[\frac{L}{h}\right] \right\}.
\tag{4}
$$

The binomial coefficients can be calculated as,

$$
\beta_j = \left(1 - \frac{a+q}{j} \right) \beta_{j-1}.
\tag{5}
$$

Below the 3D fractional-order Chua's circuit general form is given.

$$
\begin{aligned}
D^q &= A(x,y,z,t), \\
D^q &= B(x,y,z,t), \\
D^q &= C(x,y,z,t).
\end{aligned}
\tag{6}
$$

The discretization method given in (7) is used to solve the system (6) numerically with GL method.

$$
\begin{aligned}
x(t_k) &= A(x(t_{k-1}),y(t_{k-1}),z(t_{k-1}))h^q - \sum_{j=0}^{N} \beta^q_j x(t_{k-j}), \\
y(t_k) &= B(x(t_{k-1}),y(t_{k-1}),z(t_{k-1}))h^q - \sum_{j=0}^{N} \beta^q_j y(t_{k-j}), \\
z(t_k) &= C(x(t_{k-1}),y(t_{k-1}),z(t_{k-1}))h^q - \sum_{j=0}^{N} \beta^q_j z(t_{k-j}),
\end{aligned}
\tag{7}
$$

where β is the binomial coefficients as given in (5). The N value is selected as L, that is the truncation window size or as k for all the used memory elements.

The definition of the autonomous fractional-order abs system (FOABS) is given below.

$$
\begin{aligned}
D^q x &= \alpha(y - g(x)), \\
D^q y &= x - y + z, \\
D^q z &= -\beta y,
\end{aligned}
\tag{8}
$$

where $g(x) = ax + bx|x|$ and $\alpha = 9.267, \beta = 14, a = -1/6, b = 1/16$.

Using (8) in (7), the discrete form of the FOABS is,

$$
\begin{aligned}
x(t_k) &= \alpha(y(t_{k-1} - g(x(t_{k-1})))h^q - \sum_{j=0}^{N} \beta^q_j x(t_{k-j}), \\
y(t_k) &= (x(t_{k-1}) - y(t_{k-1}) + z(t_{k-1}))h^q - \sum_{j=0}^{N} \beta^q_j y(t_{k-j}), \\
z(t_k) &= (-\beta y(t_{k-1}))h^q - \sum_{j=0}^{N} \beta^q_j z(t_{k-j}).
\end{aligned}
\tag{9}
$$

The N value is selected as L, that is the truncation window size or as k for all the used memory elements. When parameters $\alpha = 9.267, \beta = 12.9$ or $14, a = -1/6, b = 1/16$ and commensurate fractional-order $q = 0.995$, the FOABS exhibits chaotic oscillations for the initial values of $x(0) = -1.01, y(0) = z(0) = -0.01$ as shown in Figure 2.
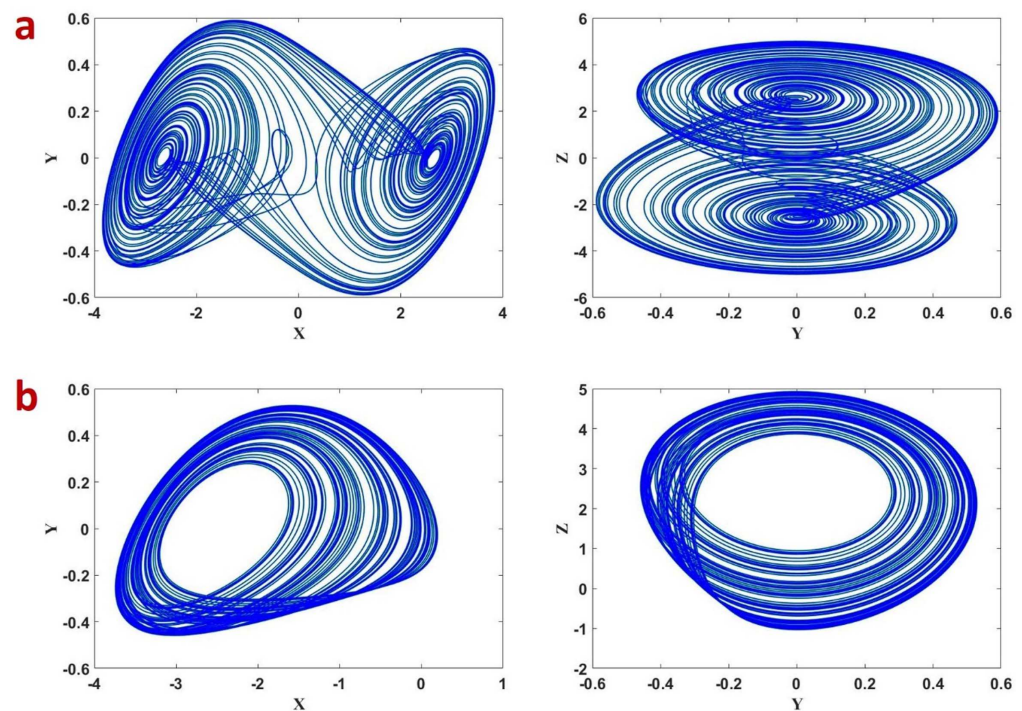
Figure 2. Phase portraits of the fractional-order abs system (FOABS) (8) for (**a**) $\beta = 12.8$; (**b**) $\beta = 14$.

4. Dynamical Properties of the FOABS System

4.1. Lyapunov Exponents

To find the Lyapunov Exponents (LEs) of the FOABS, the modified Wolf's algorithm [31,32] is used. The LEs of the FOABS are found to be $L_1 = 0.247, L_2 = 0, L_3 = -2.12$ for the initial values $x(0) = -1.01, y(0) = z(0) = -0.01$, parameter values $\alpha = 9.267, \beta = 14, a = -1/6, b = 1/16$ and fractional-order $q = 0.995$.

4.2. Route to Chaos

Bifurcation diagrams and dynamic behaviorof the FOABS are given in Figures 3 and 4 with respect to the fractional-order q and parameter β respectively. In Figure 3a the parameter $\beta = 14$ while in Figure 3b the parameter $\beta = 12$. Figure 3a shows that the FOABS exhibits chaotic behavior for $q > 0.9915$ and Figure 3b shows that the system exhibits chaotic behavior for $q > 0.953$. It is obvious that the FOABS takes period doubling route to chaos. Figure 4 shows the bifurcation with parameter β when the fractional-order $q = 0.995$, and the FOABS takes an inverse period doubling exit from chaos. For the Figures 3 and 4, the other parameters are kept at $\alpha = 9.267, a = -1/6, b = 1/16$.
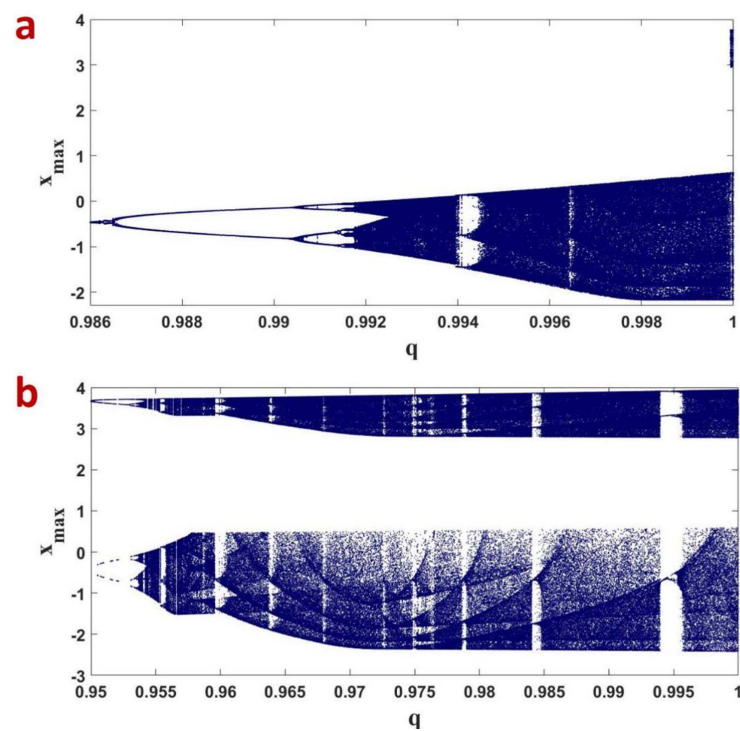
Figure 3. The bifurcation of the FOABS (8) with respect to q for (**a**) $\beta = 14$; (**b**) $\beta = 12$.
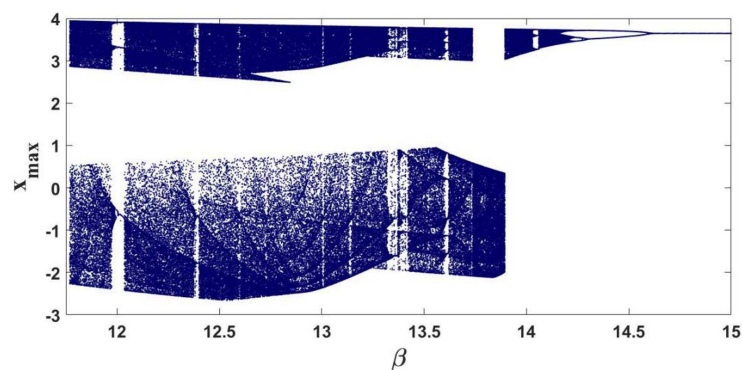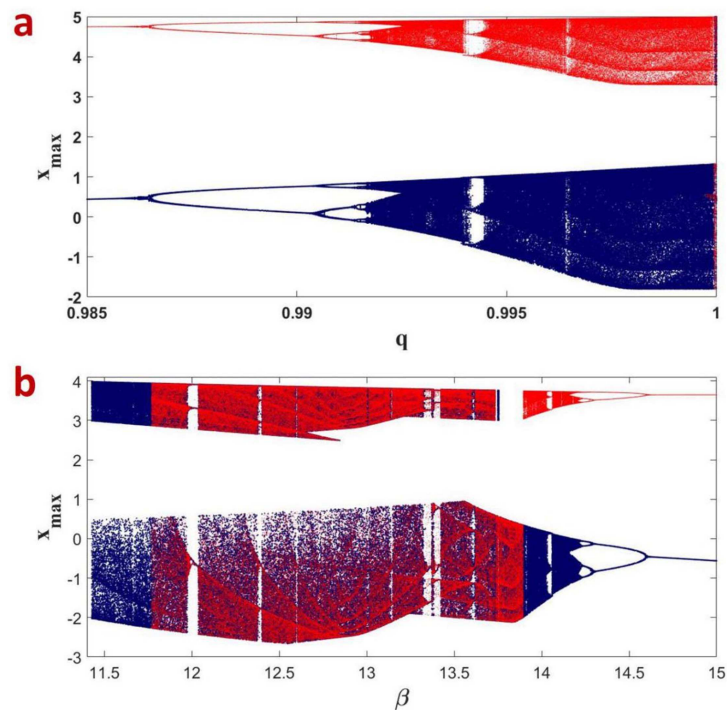

Figure 4. Bifurcation of the FOABS system (8) with respect to β for $q = 0.995$.

4.3. Multistability and Coexisting Attractors

The discontinuous bifurcations in Figures 3 and 4 confirm that the FOABS system can show multistability and coexisting attractors. To investigate this the well-known forward and backward continuation are used [33]. In forward continuation, the parameter of interests is increased from minimum to maximum and initial values are initialized to the end values of state trajectories. Finally, the local maxima of the state variables are plotted. In backward continuation, the parameter of interests is decreased from maximum to minimum. The next steps are carried out as same as in forward continuation. Figure 5a shows the multistable plots of the FOABS with fractional-order q while the other parameters kept as $\alpha = 9.267, \beta = 14, a = -1/6, b = 1/16$. Blue plots show the forward continuation and red plots show the backward continuation. Similarly Figure 5b shows the bifurcation with respect to β while other parameters kept as $\alpha = 9.267, a = -1/6, b = 1/16$. and fractional-order $q = 0.995$. Red plots show the forward continuation and blue plots show the backward continuation. Figure 6 shows the coexisting attractors for different values of q and β.

Figure 5. (**a**) Bifurcation of the FOABS with respect to q (the blue plot is forward continuation and the red plot is backward continuation); (**b**) Bifurcation of the FOABS with respect to β (the red plot is forward continuation and the blue plot is backward continuation).

$$\beta = 14, q = 0.995$$

$$\beta = 12.8, q = 0.985$$
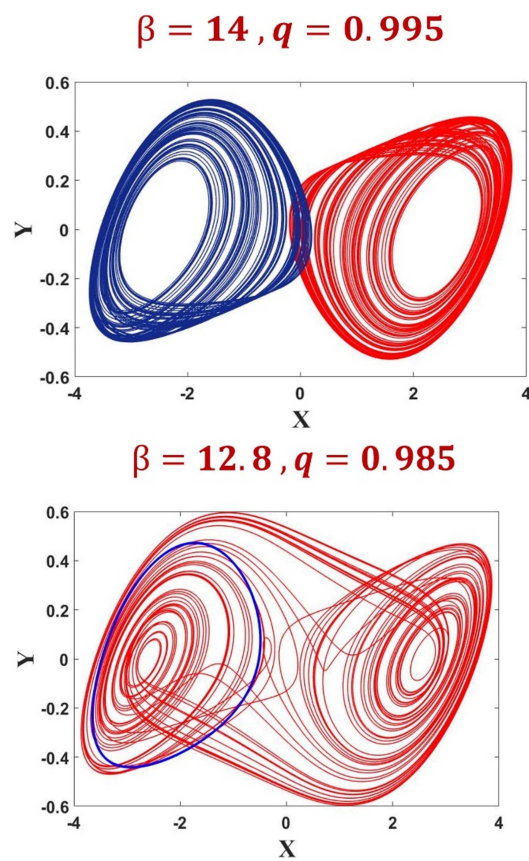
Figure 6. the coexisting attractors shown by the FOABS system (8).

5. Microcontroller-Based Implementation of FOABS System

In this section, microcontroller-based circuit design of the FOABS system was implemented at ATmega328p microcontroller in Arduino UNO board to use in real digital engineering applications. Equation (9) is adapted to the microcontroller and microcontroller program was written in the Arduino v.1.8.9 compiler program. The flow chart of the microcontroller program is given in Figure 7. In the software, the initial values $(x0, y0, z0)$, fractional-order q value and parameters (a, b, α, β) values of the FOABS system are first taken from the user via USB (Universal Serial Bus) communication with a computer. With these values, the FOAB system state variables (x, y, z) are calculated and sent via USB port.
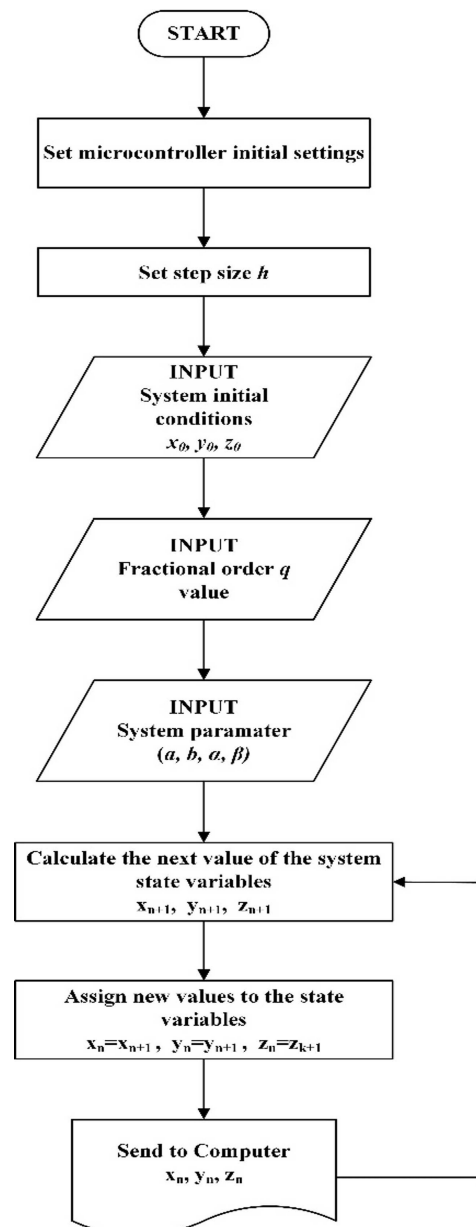

Figure 7. The flow chart of the microcontroller program.

The designed system using the Arduino–Uno microcontroller platform is connected to the computer and testing was realized in Figure 8. The test parameters were taken as follows: step size $h = 0.01, x0 = -1.01, y0 = -0.01, z0 = -0.01, q = 0.995, a = -0.16666, b = 0.0625, \alpha = 9.267$ and $\beta = 12.8$. An example screenshot of the communication between the microcontroller and the computer is given in Figure 9. The data coming from the microcontroller to the computer were drawn in Matlab (Figure 10). When Figures 2a and 10

are examined together, it is seen that the results obtained with the microcontroller and the numerical simulation results confirm each other. Desired parameters $(x0, y0, z0, q, a, b, \alpha, \beta)$ can be easily changed in the designed microcontroller-based system. In this way, the FOABS system outputs can be obtained in a very flexible way with different parameter values.

Figure 8. Microcontroller-based system test platform.

Figure 9. An example screenshot of the communication between the microcontroller and the computer.

Figure 10. 2D phase portraits of the FOABS system obtained from the microcontroller, $\beta = 12.8$ (**a**) $x - y$; (**b**) $y - z$.

6. Electronic Circuit Realization of Modified Chua's Circuit (1)

Recent years have seen rapid improvements in chaos science and also in studies on understanding chaos and chaotic systems, detecting features and differences, observing experimental data. Among such studies are chaotic circuit studies on modeling the chaotic systems [34–36]. In this section, the modeling of a chaotic system with absolute function is implemented in ORCAD-PSpice simulation environment.

The chaotic system's electronic circuit with absolute function is realized in ORCAD-PSpice. The parameter values are set as $\alpha = 9.267, \beta = 14, a = -1/6$ and $b = 1/16$ in electronic circuit implementation. Compared with the result in [16], Figure 11 has different components like inductor. In our design, there are two Op-amps for $g(x)$ function, but $g(x)$ function in [4] has four Op-amps for the circuit implementation. Select C1 = C2 = C3 = 1nF, R1 = 43 kΩ, R2 = 259 kΩ, R3 = 69 kΩ, R4 = R5 = R9 = R10 = R12 = R13 = R14 = R15 = R16 = R17 = R18 = 100 kΩ, R6 = R7 = R8 = 400 kΩ, R11 = 28.5 kΩ, R10 = 333 kΩ. The ORCAD-PSpice simulation outputs, are seen in Figures 12 and 13 with initial values $(-1.01, -0.01, -0.01)$.
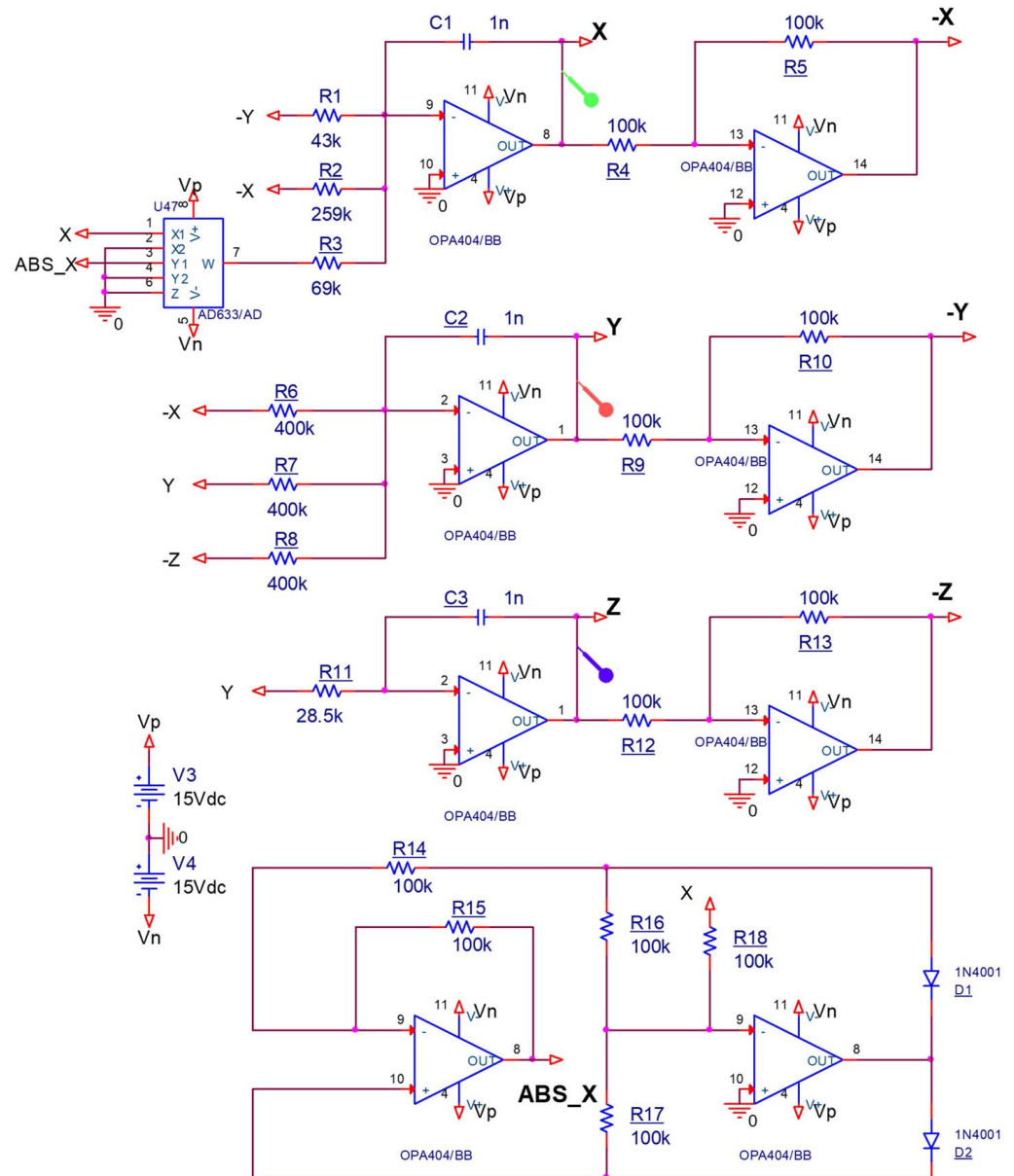
Figure 11. The circuit schematic of the electronic design for system (1).
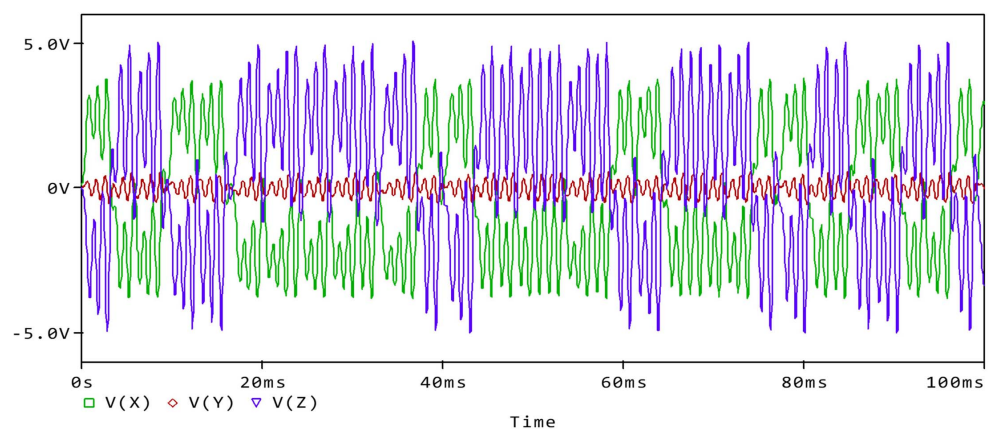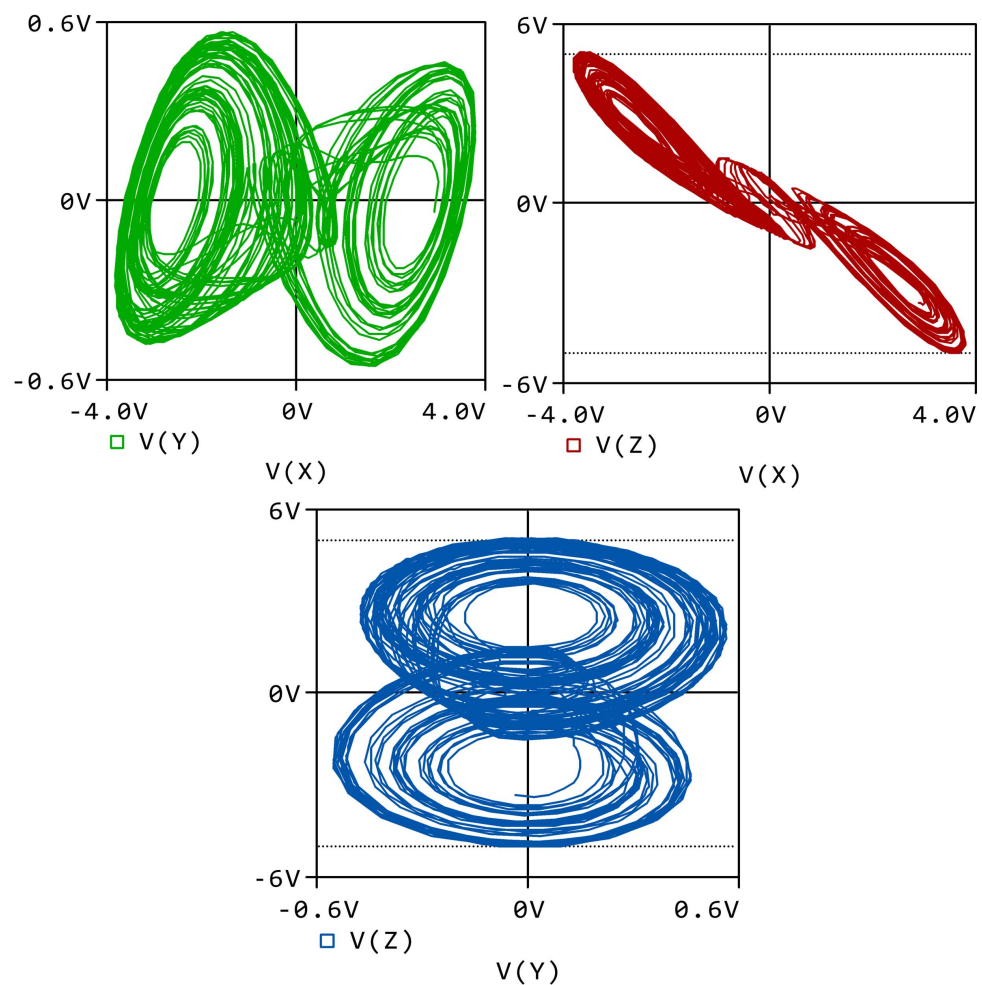

Figure 12. Time series of the system (1) state variables.

Figure 13. The all phase portraits of electronic circuit design in ORCAD-PSpice for parameters $\alpha = 9.267, \beta = 14, a = -1/6$ and $b = 1/16$ in system (1).

7. Conclusions

This paper considers the improved circuit implementation of the 3D Chua's system with this function $x|x|$. In addition, analysis and microcontroller-based implementation of dynamical properties of the fractional-order form of this modified system is presented by bifurcation diagram, phase portrait, Lyapunov exponents, multistability and coexisting attractors. Using the designed microcontroller-based circuit of the FOABS system and electronic circuit implementation, this system can be used in various chaos-based real engineering applications such as encryption, random number generator and communication. More studies on possible applications of this research are planned in future works.

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