Theoretical and computational structures on solitary wave solutions of Benjamin Bona Mahony-Burgers equation

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Abstract

This paper aims to obtain exact and numerical solutions of the nonlinear Benjamin Bona Mahony-Burgers (BBM-Burgers) equation. Here, we propose the modified Kudryashov method for getting the exact traveling wave solutions of BBM-Burgers equation and a septic B-spline collocation finite element method for numerical investigations. The numerical method is validated by studying solitary wave motion. Linear stability analysis of the numerical scheme is done with Fourier method based on von-Neumann theory. To show suitability and robustness of the new numerical algorithm, error norms L_2, L_∞ and three invariants I_1, I_2 and I_3 are calculated and obtained results are given both numerically and graphically. The obtained results state that our exact and numerical schemes ensure evident and they are penetrative mathematical instruments for solving nonlinear evolution equation.

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1 Introduction

Partial differential equations (PDEs) are extensively used as models to define numerious physical occurrences and play crucial roles in many sciences mostly in physics, applied mathematics and in engineering problems. Exact solutions of these equations are commonly not procurable, especially when the nonlinear terms are contained. But the modified Kudryashov method is a robust solution scheme to obtain the exact solutions of PDEs in mathematical physics and biology [1]. This method can be applied to non-integrable equations as well as integrable ones and it was first performed in the fractional differential equations by Ege and Misirli [2]. Recently, this method has received a significant relevance due to its capabilities in extracting new exact solutions of PDEs in integer order as well as fractional order. Some conformable fractional differential equations arising in mathematical biology [1], space-time fractional coupled EW and coupled MEW equations [3], conformable time-fractional Klein–Gordon equations [4], fractional version of the variety of Boussinesq-like equations [5] have been solved analiytically with modified Kudryashov method. The BBM (Benjamin Bona Mahoney) equation,

$$U_t + U_x + aUU_x - bU_{xxt} = 0, \tag{1}$$

is one of the prototype PDEs of the nonlinear dispersive waves which has many implementation in several areas such as the nonlinear transverse waves in shallow water, ion-acoustic waves in plasma,

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magnetohydromagnetic wave in cold plasma, acoustic-gravity waves in compressible fluids, pressure waves in liquid–gas bubbles and acoustic waves inharmonic crystals. The solutions of this equation are types of solitary waves named as solitons whose forms are not changed after the collision. It was first propounded as a model for small-amplitude long-waves on the surface of water in a channel by Peregrine [6, 7] and widely investigated by Benjamin et al. [8]. An exact solution of the equation was obtained under the limited initial and boundary conditions in [9] so it got fascinate from a numerical point of view. Therefore, numerical solutions of the BBM equation have been the matter of many papers. A variety of numerical methods in name including finite difference [10, 11, 12, 13], pseudo-spectral [14] , meshfree method [15], Adomian decomposition method [16] and various forms of finite element methods in [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] have been used for the solution of the BBM equation. Also the Benjamin-Bona- Mahoney-Burgers (BBM-Burgers) equation has been numerically discussed and examined by many authors. Quadratic B-spline finite element method for the spatial variable combined with a Newton method for the time variable is suggested to approximate solution of BBM-Burgers equation by Y.-X. Yin and G.-Rı Pıao [28]. A quadratic, cubic and quartic B-spline collocation methods can be found in [29, 30, 31, 32], respectively. A finite element model for the BBM-Burgers equation with a high-order dissipative term based on adaptive moving meshes is proposed by Lu et al. [33]. The hybrid BBM-Burgers equation with dual power-law nonlinearity is studied in [34]. Numerical solutions of the BBM-Burgers equation in one space dimension have analyzed using Crank-Nicolson-type finite difference method in [35]. The asymptotic behavior of solutions to the Cauchy problem for the BBM-Burgers equation have considered by Mei and Schmeiser [36]. C. Kondo and C. M. Weblere [37] have studied the existence and convergence of the smooth solutions of the generalized BBM-Burgers equation. The (G'/G) expansion method has been implemented to the equation in [38, 39, 40]. A. Mohebbi and Z. Faraz [41] have investigated the solitary wave solution of nonlinear BBM-Burgers equation using a high-order linear finite difference scheme. Solitary-wave solutions of the nonlinear Benjamin–Bona–Mahony–Burgers(BBM–Burgers) equation based on a lumped Galerkin technique using cubic B-spline finite elements for the spatial approximation have been studied by S. B. G. Karakoc and S. K. Bhowmik [42]. A family of local fractional two-dimensional Burgers-type equations (2DBEs) and the local fractional Riccati differential equation method are proposed in the article for the first time by X-J. Yang, F. Gao and H. M. Srivastava [43].

This present study is encouraged by a wish to expand the studies made in the literature concerned with the BBM-Burgers equation. The modified Kudryashov method, developed recently, is a convenient and an effective method for getting exact solutions of nonlinear evolution equations. Present work is divided into eight sections and the organization is as follows: In Section 2, the governing equation is introduced, septic B-spline functions to be used are expressed and collocation finite element technique has been practiced to BBM-Burgers equation. A linear stability analysis of the numerical scheme is explored in Section 3 followed by Section 4 which belongs to numerical experiments of traveling single solitary wave with different initial and boundary conditions. Description of the modified Kudryashov method and exact solutions of the BBM-Burgers equation are given in Section 5 and 6, respectively. Graphical illustrations of the exact solutions are plotted in Section 7. Finally, conclusion is drawn in Section 8 followed by references.

2 The governing equation and analysis of the method

A mathematical form of progression of small-amplitude long waves in nonlinear dispersive media is defined by the following BBM-Burgers equation:

$$U_t - U_{xxt} - \alpha U_{xx} + \beta U_x + UU_x = 0, \qquad a \le x \le b, \qquad 0 \le t \le T, \tag{2}$$

with an initial condition

$$U(x,0) = f(x), \qquad a \le x \le b, \tag{3}$$

and boundary conditions

$$\begin{array}{ll} U(a,t)=0, & U(b,t)=0, \\ U_x(a,t)=0, & U_x(b,t)=0, \\ U_{xx}(a,t)=0, & U_{xx}(b,t)=0, \quad t>0 \end{array} \tag{4}$$

where α and β are positive constants. If α is taken zero in Eq.(2), Eq.(1) is obtained. BBM-Burgers equation comprises both dispersive and dissipative influences. The dissipative term is $-\alpha U_{xx}$ and its dissipative influence is the same as the following Burgers equation

$$U_t - \alpha U_{xx} + \beta U_x + UU_x = 0. \tag{5}$$

The dispersive influence of Eq.(2) is the same as the Eq.(1) in so far as dispersive term $-U_{xxt}$. To obtain the solution on interval $a \le x \le b$ division $a = x_0 < x_1 < ... < x_N = b$ of the space domain is imagined scattered uniformly with $h = \frac{b-a}{N}$. The septic B-spline functions $\varphi_m(x)$ $(m = -3, -2, ..., N+2, N+3)$ at the nodes x_m which form a basis for functions, described on the solution zone $[a, b]$ by Prenter [44]

$$\varphi_m(x) = \frac{1}{h^7} \begin{cases} (x-x_{m-4})^7 & [x_{m-4}, x_{m-3}] \\ (x-x_{m-4})^7 - 8(x-x_{m-3})^7 & [x_{m-3}, x_{m-2}] \\ (x-x_{m-4})^7 - 8(x-x_{m-3})^7 + 28(x-x_{m-2})^7 & [x_{m-2}, x_{m-1}] \\ (x-x_{m-4})^7 - 8(x-x_{m-3})^7 + 28(x-x_{m-2})^7 - 56(x-x_{m-1})^7 & [x_{m-1}, x_m] \\ (x_{m+4}-x)^7 - 8(x_{m+3}-x)^7 + 28(x_{m+2}-x)^7 - 56(x_{m+1}-x)^7 & [x_m, x_{m+1}] \\ (x_{m+4}-x)^7 - 8(x_{m+3}-x)^7 + 28(x_{m+2}-x)^7 & [x_{m+1}, x_{m+2}] \\ (x_{m+4}-x)^7 - 8(x_{m+3}-x)^7 & [x_{m+2}, x_{m+3}] \\ (x_{m+4}-x)^7 & [x_{m+3}, x_{m+4}] \\ 0 & otherwise. \end{cases} \tag{6}$$

The numerical treatment for the BBM-Burgers equation utilizing the collocation method with septic B-spline is to get an approximate solution $U_N(x,t)$ to the exact solution $U(x,t)$ given by

$$U_N(x,t) = \sum_{m=-3}^{N+3} \varphi_m(x)\delta_m(t) \tag{7}$$

where $\delta_m(t)$ are time dependent coefficients. Each septic B-spline covers eight elements, so each element $[x_m, x_{m+1}]$ is covered by eight B-splines. A spesific finite interval $[x_m, x_{m+1}]$ is planned to the interval $[0, 1]$ by a local coordinate transformation defined by $h\xi = x - x_m$, $0 \le \xi \le 1$. So septic B-splines (6) in terms of ξ over $[0, 1]$ can be given as follows:

$$
\begin{aligned}
\varphi_{m-3} &= 1-7\xi+21\xi^2-35\xi^3+35\xi^4-21\xi^5+7\xi^6-\xi^7,\\
\varphi_{m-2} &= 120-392\xi+504\xi^2-280\xi^3+84\xi^5-42\xi^6+7\xi^7,\\
\varphi_{m-1} &= 1191-1715\xi+315\xi^2+665\xi^3-315\xi^4-105\xi^5+105\xi^6-21\xi^7,\\
\varphi_{m} &= 2416-1680\xi+560\xi^4-140\xi^6+35\xi^7,\\
\varphi_{m+1} &= 1191+1715\xi+315\xi^2-665\xi^3-315\xi^4+105\xi^5+105\xi^6-35\xi^7,\\
\varphi_{m+2} &= 120+392\xi+504\xi^2+280\xi^3-84\xi^5-42\xi^6+21\xi^7,\\
\varphi_{m+3} &= 1+7\xi+21\xi^2+35\xi^3+35\xi^4+21\xi^5+7\xi^6-\xi^7,\\
\varphi_{m+4} &= \xi^7.
\end{aligned} \tag{8}
$$

For this problem, the finite elements are described with the space $[x_m, x_{m+1}]$. Using Eq.(6) and Eq.(7), the nodal values of U_m, U'_m, U''_m, U'''_m and U^{iv}_m are given in terms of the element parameters δ_m with

$$
\begin{aligned}
U_N(x_m,t) = U_m &= \delta_{m-3}+120\delta_{m-2}+1191\delta_{m-1}+2416\delta_m+1191\delta_{m+1}+120\delta_{m+2}+\delta_{m+3},\\
U'_m &= \tfrac{7}{h}(-\delta_{m-3}-56\delta_{m-2}-245\delta_{m-1}+245\delta_{m+1}+56\delta_{m+2}+\delta_{m+3}),\\
U''_m &= \tfrac{42}{h^2}(\delta_{m-3}+24\delta_{m-2}+15\delta_{m-1}-80\delta_m+15\delta_{m+1}+24\delta_{m+2}+\delta_{m+3}),\\
U'''_m &= \tfrac{210}{h^3}(-\delta_{m-3}-8\delta_{m-2}+19\delta_{m-1}-19\delta_{m+1}+8\delta_{m+2}+\delta_{m+3}),\\
U^{iv}_m &= \tfrac{840}{h^4}(\delta_{m-3}-9\delta_{m-1}+16\delta_m-9\delta_{m+1}+\delta_{m+3})
\end{aligned} \tag{9}
$$

and the variation of U over the element $[x_m, x_{m+1}]$ is given by

$$
U = \sum_{m=-3}^{N+3} \varphi_m \delta_m. \tag{10}
$$

Substituting the approximate solution (7) and putting the nodal values of (10) and its derivatives given by (9) into Eq.(2) yields the following set of ordinary differential equations of the form

$$
\begin{aligned}
&\dot{\delta}_{m-3}+120\dot{\delta}_{m-2}+1191\dot{\delta}_{m-1}+2416\dot{\delta}_m+1191\dot{\delta}_{m+1}+120\dot{\delta}_{m+2}+\dot{\delta}_{m+3}\\
&-\tfrac{42}{h^2}(\dot{\delta}_{m-3}+24\dot{\delta}_{m-2}+15\dot{\delta}_{m-1}-80\dot{\delta}_m+15\dot{\delta}_{m+1}+24\dot{\delta}_{m+2}+\dot{\delta}_{m+3})\\
&-\tfrac{42}{h^2}(\delta_{m-3}+24\delta_{m-2}+15\delta_{m-1}-80\delta_m+15\delta_{m+1}+24\delta_{m+2}+\delta_{m+3})\\
&+\tfrac{7}{h}Z_m(-\delta_{m-3}-56\delta_{m-2}-245\delta_{m-1}+245\delta_{m+1}+56\delta_{m+2}+\delta_{m+3})=0,
\end{aligned} \tag{11}
$$

where

$Z_m = U_m = (\delta_{m-3}+120\delta_{m-2}+1191\delta_{m-1}+2416\delta_m+1191\delta_{m+1}+120\delta_{m+2}+\delta_{m+3})$.

After discretizing the time derivative by the usual inite difference aproximation

$$
\dot{\delta}_i = \frac{\delta_i^{n+1}-\delta_i^n}{\Delta t} \tag{12}
$$

and the spatial variables and their derivatives by Crank-Nicolson formula

$$
\delta_i = \frac{\delta_i^{n+1}+\delta_i^n}{2}, \tag{13}
$$

in Eq.(11), we derive a repetition relationship between two time levels n and $n+1$ relating two unknown parameters δ_i^{n+1}, δ_i^n for $i = m-3, m-2, ..., m+2, m+3$

$$
\begin{aligned}
&\gamma_1\delta_{m-3}^{n+1}+\gamma_2\delta_{m-2}^{n+1}+\gamma_3\delta_{m-1}^{n+1}+\gamma_4\delta_{m}^{n+1}+\gamma_5\delta_{m+1}^{n+1}+\gamma_6\delta_{m+2}^{n+1}+\gamma_7\delta_{m+3}^{n+1}\\
&=\gamma_7\delta_{m-3}^{n}+\gamma_6\delta_{m-2}^{n}+\gamma_5\delta_{m-1}^{n}+\gamma_4\delta_{m}^{n}+\gamma_3\delta_{m+1}^{n}+\gamma_2\delta_{m+2}^{n}+\gamma_1\delta_{m+3}^{n},
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
&\gamma_1=[1-E-M-K(1+Z_m)],\\
&\gamma_2=[120-24E-24M-56K(1+Z_m)],\\
&\gamma_3=[1191-15E-15M-245K(1+Z_m)],\\
&\gamma_4=[2416+80E+80M],\\
&\gamma_5=[1191-15E-15M+245K(1+Z_m)],\\
&\gamma_6=[120-24E-24M+56K(1+Z_m)],\\
&\gamma_7=[1-E-M+K(1+Z_m)],\\
&m=0,1,\ldots,N,\qquad E=\tfrac{42}{h^2},\qquad M=\tfrac{21\Delta t}{h^2},\qquad K=\tfrac{7}{2h}\Delta t.
\end{aligned}
\tag{15}
$$

In this way, the system (14) involves of $(N+1)$ linear equations containing $(N+7)$ unknown coefficients $(\delta_{-3},\delta_{-2},\delta_{-1},\ldots,\delta_{N+1},\delta_{N+2},\delta_{N+3})$. So, we require six additional equations corresponding to the unknowns $\delta_{-3},\delta_{-2},\delta_{-1},\ldots,\delta_{N+1},\delta_{N+2}$ and δ_{N+3} to obtain a unique solution for this resulting system. The six additional equations are obtained from the boundary conditions (4). After eliminating $\delta_{-3},\delta_{-2},\delta_{-1}$,$\delta_{N+1},\delta_{N+2}$ and δ_{N+3}, the system (14) is reduced into a matrix system of $(N+1)$ linear equations as

$$
R\mathbf{d^{n+1}}=S\mathbf{d^{n}}. \tag{16}
$$

Two or three inner iterations are implemented to the term $\delta^{n*}=\delta^n+\frac{1}{2}(\delta^n-\delta^{n-1})$ at each time step to overcome the non-linearity caused by Z_m. Before the beginning of the solution procedure, initial parameters d^0 are established by using the initial condition and following derivatives at the boundaries;

$$
\begin{aligned}
U_N(x,0)&=U(x_m,0); && m=0,1,2,...,N && (17)\\
(U_N)_x(a,0)&=0, && (U_N)_x(b,0)=0, && (18)\\
(U_N)_{xx}(a,0)&=0, && (U_N)_{xx}(b,0)=0, && (19)\\
(U_N)_{xxx}(a,0)&=0, && (U_N)_{xxx}(b,0)=0. && (20)
\end{aligned}
$$

Therefore we obtaine the following matrix form for the initial vector d^0;

$$
Vd^0=W, \tag{21}
$$

$$
\text{where}\quad V=\begin{bmatrix}
1536 & 2712 & 768 & 24 & & & & & & \\
\frac{82731}{81} & \frac{210568.5}{81} & \frac{104796}{81} & \frac{10063.5}{81} & 1 & & & & & \\
\frac{9600}{81} & \frac{96597}{81} & \frac{195768}{81} & \frac{96474}{81} & 120 & 1 & & & & \\
 & & \ddots & & & & & & & \\
 & & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1\\
 & & & & 1 & 120 & \frac{96474}{81} & \frac{195768}{81} & \frac{96597}{81} & \frac{9600}{81}\\
 & & & & & 1 & \frac{10063.5}{81} & \frac{104796}{81} & \frac{210568.5}{81} & \frac{82731}{81}\\
 & & & & & & 24 & 768 & 2712 & 1536
\end{bmatrix}
$$

$d^0=(\delta_0,\delta_1,\delta_2,...,\delta_{N-2},\delta_{N-1},\delta_N)^T$ and $W=(U(x_0,0),U(x_1,0),...,U(x_{N-1},0),U(x_N,0))^T$.

3 Stability analysis

The stability of the presented technique is explored by practicing Fourier method based on Von-Neumann theory. Presuming that the cardinality U in the nonlinear term UU_x is locally fixed. Placementing the Fourier mode $\delta_m^n = \xi^n e^{i\sigma mh}$,$(i=\sqrt{-1})$ into the form of (14) we obtain,

$$\begin{aligned} & \xi^{n+1}(\gamma_1 e^{i(m-3)\theta} + \gamma_2 e^{i(m-2)\theta} + \gamma_3 e^{i(m-1)\theta} + \gamma_4 e^{im\theta} + \gamma_5 e^{i(m+1)\theta} + \gamma_6 e^{i(m+2)\theta} + \gamma_7 e^{i(m+3)\theta}) \\ = \; & \xi^{n}(\gamma_7 e^{i(m-3)\theta} + \gamma_6 e^{i(m-2)\theta} + \gamma_5 e^{i(m-1)\theta} + \gamma_4 e^{im\theta} + \gamma_3 e^{i(m+1)\theta} + \gamma_2 e^{i(m+2)\theta} + \gamma_1 e^{i(m+3)\theta}) \end{aligned} \tag{22}$$

where σ is mode number, h is the element size, $\theta = \sigma h$. If we simplify the Eq.(22),

$$\xi = \frac{A+iB}{A-iB} \tag{23}$$

is obtained where

$$\begin{aligned} & A = (2382 - 30E - 30M)\cos(\theta) + (240 - 48E - 48M)\cos(2\theta) + (2 - 2E + 2M)\cos(3\theta) + \\ & (2416 + 80E + 80M), \\ & B = (490K(1+Z_m))\sin(\theta) + (112K(1+Z_m))\sin(2\theta) + (2K(1+Z_m))\sin(3\theta) + \\ & (2416 + 80E - 80M). \end{aligned} \tag{24}$$

According to the Fourier stability analysis, for the given scheme in order to be stable, the condition $|\xi| \leq 1$ must be satisfied. Using a symbolic programming software or using simple calculations, since $a^2 + b^2 = a^2 + (-b)^2$ it becomes evident that the modulus of $|\xi|$ is 1. Therefore the linearized algorithm is unconditionally stable.

4 Numerical results and discussion

In this part, in order to verify our numerical algorithm, we take into consideration some experiments involving: Dispersion of single solitary waves, interaction of two and three solitary. For these three problems, to demonstrate how suitable our numerical algorithm foresees the position and amplitude of the solution as the simulation progresses, we handle the following error norms:

$$L_2 = \left\|U^{exact} - U_N\right\|_2 \simeq \sqrt{h\sum_{J=0}^{N}\left|U_j^{exact} - (U_N)_j\right|^2}, \tag{25}$$

and

$$L_\infty = \left\|U^{exact} - U_N\right\|_\infty \simeq \max_j \left|U_j^{exact} - (U_N)_j\right|. \tag{26}$$

There are three conserved quantities for the BBM-Burgers equation. These are correspond to mass, momentum, energy and given by

$$I_1 = \int_{-\infty}^{\infty} U(x,t)dx, \quad I_2 = \int_{-\infty}^{\infty} [U^2(x,t) + U_x^2(x,t)]dx, \quad I_3 = \int_{-\infty}^{\infty} [U^3(x,t) + 3U^2(x,t)]dx \tag{27}$$

respectively.

4.1 Dispersion of a single solitary wave

To confirm our numerical scheme, we take into consideration two cases.

Case 1. For this case, we will suggest some numerical results for the BBM equation (1) which, as indicated in the introduction, can be derived from the BBM-Burgers equation (2) by taking $\alpha = 0$ and $\beta = 1$. We tackle Eq.(1) with the boundary conditions $U \to 0$ as $x \to \pm\infty$ and the initial condition

$$U(x,0) = 3c \sec h^2[k(x - x_0)]. \tag{28}$$

This problem has the following theoretical solitary wave solution

$$U(x,t) = 3c \sec h^2[k(x - x_0 - vt)], \tag{29}$$

where $v = 1 + \varepsilon c$ is the wave velocity and $k = \frac{1}{2}\sqrt{\frac{\varepsilon c}{\mu(1+\varepsilon c)}}$. This equation stands for a single soliton of amplitude $3c$ with the constant speed $1 + \varepsilon c$ and initially centred on x_0. The values of the parameters are taken firstly, $c = h = \Delta t = 0.1$ and secondly $c = 0.03$, $h = \Delta t = 0.1$ over the interval $[-40, 60]$ to match up with that of previous papers. The exact values of the invariants are found as $I_1 = 3.9799497$, $I_2 = 0.81046249$ and $I_3 = 2.579007$ for $c = 0.1$ and $I_1 = 2.1094074$, $I_2 = 0.127302$ and $I_3 = 0.388806$ for $c = 0.03$. The simulations are performed to time $t = 20$ to obtain the error norms and three conserved quantities. The obtained datas for different values of c have been given in Table (1) and (2). These tables clearly show that the error norms obtained by our method are less than the others and our invariants are almost constant as time increases. It is noticeably seen from the tables that for $c = 0.1$; the invariant I_1 changes from its initial value by less than 4.31×10^{-5} whereas the change of invariants I_2 and I_3 are zero and for $c = 0.03$; I_1, I_2, I_3 change from their initial value by less than 2.41×10^{-3}, 9×10^{-7} and 2.2×10^{-6}, respectively. Also, the changes of the invariants agree with their exact values. We have found out error norms L_2 and L_∞ are obtained sufficiently small during the computer run. Therefore we can say our method is sensibly conservative. Fig. (1) shows the solutions at $t = 0, 10$ and 20. As seen, single solitons move to the right at a constant speed and preserves its amplitude and shape with increasing time as anticipated. Initially, for $c = 0.1$, the amplitude of solitary wave is 0.30000 and its top position is pinpionted at $x = 0$. At $t = 20$ its amplitude is noted as 0.29997 with center $x = 22$ and for $c = 0.03$, the amplitude of solitary wave is 0.08999 and its top position is pinpionted at $x = 0$. At $t = 20$ its amplitude is noted as 0.08999 with center $x = 20.6$. Thereby the absolute difference in amplitudes over the time interval $[0, 20]$ are observed as 3×10^{-5} and 0, respectively. The quantile of error at disjoint times are depicted in Fig.(2) for $c = 0.1$ and 0.03, respectively. The error aberration varies from -8×10^{-5} to 1×10^{-4} for $c = 0.1$ and from -4×10^{-4} to 4×10^{-4} for $c = 0.03$.

Case 2. In this case, we conceive the Eq.(2) with $\alpha = 0, \beta = 1$ and the boundary conditions $U \to 0$ as $x \to \pm\infty$, the initial condition

$$U(x,0) = \sec h^2[\frac{x}{4}]. \tag{30}$$

For this case, the exact solution of this problem is

$$U(x,t) = \sec h^2[\frac{x}{4} - \frac{t}{3}]. \tag{31}$$

TABLE 1. Invariants and error norms for the single solitary wave with $c = h = \Delta t = 0.1$ over the region $[-40, 60]$ for different times.

Method	*Time*	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Collocation Septic	0	3.9799264	0.8104627	2.5790082	0	0
	4	3.9799532	0.8104627	2.5790082	0.0459217	0.0179172
	8	3.9799717	0.8104627	2.5790082	0.0903127	0.0361521
	12	3.9799860	0.8104627	2.5790082	0.1329449	0.0530899
	16	3.9799910	0.8104627	2.5790082	0.1730352	0.0680887
	20	3.9799695	0.8104627	2.5790082	0.2113133	0.0818479
$h = 0.05$	20	3.9799843	0.8104621	2.5790061	0.2193390	0.0823461
$h = 0.01$	20	3.9800054	0.8104616	2.5790045	0.2346357	0.0876049
Galerkin quadratic ($h = 0.1$) [23]	20	3.97989	0.810467	2.57902	0.220	0.086
Finite difference ($h = 0.1$) [23]	20	4.41219	0.897342	2.85361	196.1	67.35
[24]	20	3.98203	0.808650	2.57302	4.688	1.755
[18]	20	3.96160	0.804185	2.55829	0.018	1.566
[20]	20	3.98206	0.811164	2.58133	0.511	0.198
[19]	20	3.97988	0.810276	2.57839	0.30	0.116
[21]	20	3.97988	0.810465	2.57901	0.219	0.086
[29]	20	3.97988	0.810461	2.579	0.307172	0.117734
[32]	20	-	-	-	0.20	0.078

TABLE 2. Invariants and error norms for the single solitary wave with $c = 0.03, h = 0.1$, $\Delta t = 0.1$ over the region $[-40, 60]$ for different times.

Method	*Time*	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Collocation Septic	0	2.1070646	0.1273018	0.3888045	0	0
	4	2.1084334	0.1273019	0.3888061	0.15006771	0.19652671
	8	2.1093708	0.1273021	0.3888070	0.30962660	0.29397281
	12	2.1099722	0.1273023	0.3888075	0.44953010	0.34225072
	16	2.1101033	0.1273025	0.3888077	0.55766109	0.36615936
	20	2.1094796	0.1273027	0.3888067	0.65311575	0.41868081
$h = 0.05$	20	2.1105225	0.1273030	0.3888079	0.88555502	0.41872161
$h = 0.125$	20	2.1091488	0.1273027	0.3888064	0.58532273	0.41865389
[29]	20	2.10460	0.127302	0.388802	0.562458	0.431512
[31]	20	-	-	-	9.40151	3.54203

For the calculation, we choose the different space and time steps and the run of the algorithm is carried up to time $t = 40$ over the problem domain $[-40, 100]$. The error norms L_2, L_∞ and conservation quantities I_1, I_2 and I_3 are computed, which are given in the Table (3) coupled with the results of the previous methods for comparison. This table clearly show that the error norms procured by our method are less than the others and our invariants are almost constant as time increases. It is noticeably seen from the tables that the invariant I_1 and I_3 change from their initial values by less than 1×10^{-7} whereas the change of invariants I_2 are zero. Also, the changes of the invariants agree with their analytic values. We have found out error norms L_2 and L_∞ are obtained sufficiently small during the computer run. Therefore we can say our method is sensibly conservative. Fig.(3) shows the solutions at $t = 0, 5, 10, 15, 20, 25, 30, 35$ and 40. As seen, single solitons move to the right at a constant speed and conserves its amplitude and shape with increasing time as expected. Initially for $h = 0.05$ and $\Delta t = 0.025$, the amplitude of solitary wave is 0.99999 and its top position is pinpionted at $x = 0$. At $t = 40$ its amplitude is noted as 0.99995

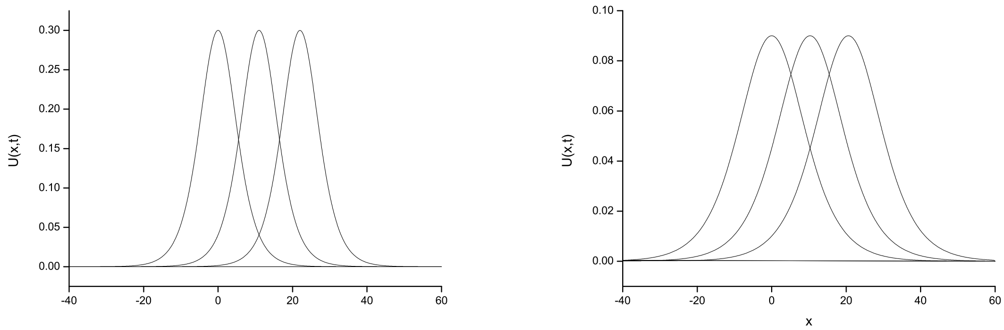

FIGURE 1. Single solitary wave with $c = h = \Delta t = 0.1$ and $c = 0.03$, $h = \Delta t = 0.1$ over the $-40 \leq x \leq 60$ at $t = 0, 10$ and 20.

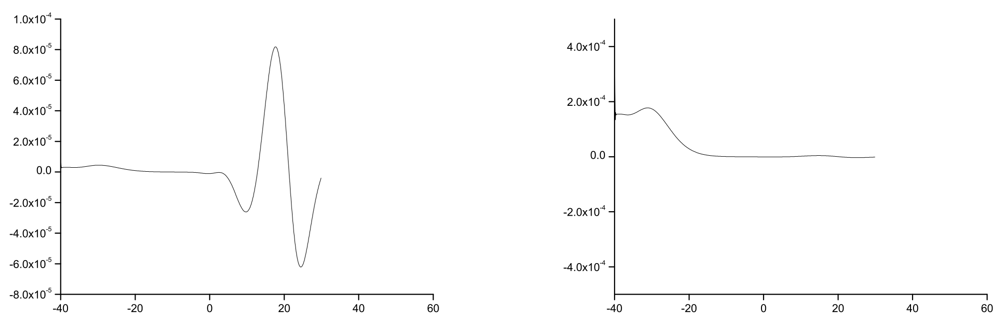

FIGURE 2. Error distribution at $t = 20$ for the parameters $c = h = \Delta t = 0.1$ and $c = 0.03$, $h = \Delta t = 0.1$ over the $-40 \leq x \leq 60$.

with center $x = 53.35$. Therefore the absolute difference in amplitudes over the time interval $[0, 40]$ are found as 4×10^{-5} and 0, respectively. The aberration of error at discrete times are designed in Fig.(4). The error deviation varies from -3×10^{-4} to 3×10^{-4} .

5 Description of the modified Kudryashov method

To illustrate the basic ideas of our method, consider the following nonlinear differential equations

$$F\left(u,\ D_t u,\ D_x u,\ D_{xx} u,\ D_{xxt} u,\ \ldots\right). \tag{32}$$

Applying the transformation

$$\begin{aligned} u\left(x,\ t\right) &= f\left(\xi\right), \\ \xi &= k\, x - c\, t - x_0, \end{aligned} \tag{33}$$

TABLE 3. Invariants and error norms for the single solitary wave with $h = 0.05$, $\Delta t = 0.025$ over the region $[-40, 100]$ for different times.

Method	*Time*	L_2	L_∞	I_1	I_2	I_3
Our $h = 0.2$, $\Delta t = 0.4$	10	0.03187463	0.01469148	8.0000009	5.5999998	20.2663292
	20	0.05466544	0.02323583	8.0000010	5.5999998	20.2660683
	30	0.07362302	0.03019023	8.0000010	5.5999998	20.2659488
	40	0.09120663	0.03668291	8.0000010	5.5999998	20.2658946
[41]	40	-	0.10976282			
Our $h = \Delta t = 0.1$	10	0.00202237	0.00093646	8.0000021	5.6000010	20.2666693
	20	0.00346138	0.00147827	8.0000021	5.6000010	20.2666683
	30	0.00472296	0.00193615	8.0000022	5.6000010	20.2666676
	40	0.00595841	0.00239601	8.0000022	5.6000010	20.2666676
[41]	40	-	0.00747237			
Our $h = 0.05$, $\Delta t = 0.025$	10	0.00011498	0.00005449	7.9999964	5.6000010	20.2666706
	20	0.00027249	0.00010719	7.9999965	5.6000010	20.2666705
	30	0.00045131	0.00017379	7.9999965	5.6000010	20.2666705
	40	0.00062752	0.00023925	7.9999966	5.6000010	20.2666705
[41]	40	-	0.00046983			
Our $h = 0.2$, $\Delta t = 0.01$	10	0.00002319	0.00001037	8.0000009	5.5999998	20.2666659
	20	0.00001037	0.00001147	8.0000010	5.5999998	20.2666659
	30	0.00002208	0.00000752	8.0000010	5.5999998	20.2666659
	40	0.00002593	0.00001052	8.0000010	5.5999998	20.2666659
[32]	20	0.00060007	0.00031641			

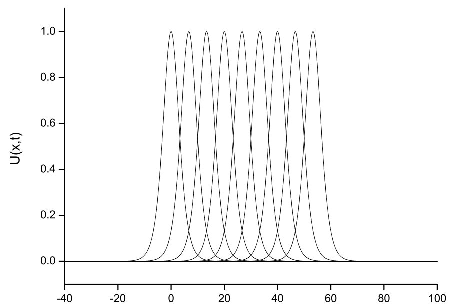

FIGURE 3. Single solitary wave with $h = 0.05$, $\Delta t = 0.025$ over the $-40 \leq x \leq 100$ at $t = 0, 5, ..., 40$.

where k and c are nonzero constants and x_0 is arbitrary constant, converts (32) into an integer order nonlinear ordinary differential equations as follows

$$H(f', f'', f''', \ \dots \ ) \ = 0, \tag{34}$$

where the derivatives are with respect to ξ. It is assumed that the solutions of (34) is presented as a finite series, say

$$f\left(\xi\right) = \sum_{n=0}^{N} a_n \, Q^n\left(\xi\right), \tag{35}$$

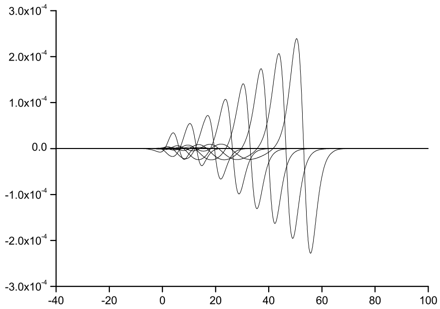

FIGURE 4. Error distribution for the parameters $h = 0.05, \Delta t = 0.025$ over the $[-40, 100]$.

where $a_n,\ b_n,\ n = 0,\ 1,\ 2,\ \ldots\ldots, N(a_N, b_N \neq 0)$are constants can be computed, and $Q^n\left(\xi\right)$is the following function.

$$Q\left(\xi\right) = \frac{1}{1 + d\ m^{\xi}}, \tag{36}$$

which satisfies the first-order equation

$$Q'\left(\xi\right) = \left(Q^2\left(\xi\right) - Q\left(\xi\right)\right)\ \ \log\ (m)\,. \tag{37}$$

It should be mentioned that the value of N is usually determined by balancing the linear and nonlinear terms of highest orders in (34). Substituting Eq.(35) and its necessary derivatives, for example

$$\begin{aligned} f' &= \textstyle\sum_{n=1}^{N} a_n\, n\, Q^n\left(Q - 1\right)\ \ \log\ (m)\,, \\ f'' &= \textstyle\sum_{n=1}^{N} a_n\, n\ Q^n\left(Q - 1\right)\ \ \left(\left(1 + n\right)Q - n\right)\ \ \left(\log\ (m)\right)^2, \end{aligned} \tag{38}$$

into (34) gives

$$P\left(Q\left(\xi\right)\right) = 0, \tag{39}$$

where $P\left(Q\left(\xi\right)\right)$ is a polynomial in $Q\left(\xi\right)$. By equating the coefficient of each power of $Q\left(\xi\right)$ in (35) to zero, a system of algebraic equations will be obtained whose solution yields the exact solutions of (32).

6 Exact solutions of the BBM-Burgers equation

Applying the wave transformation (33), we can reduce (32) to the nonlinear ordinary differential equation as the following

$$-c\, f' + \ ck^2\, f''' - \alpha\, k^2\ f'' + \beta\ k\ f' + k(\,f^2)^{'} = 0. \tag{40}$$

Integrating (40) once with respect to ξ,yields

$$-c\, f + \ ck^2\, f'' - \alpha\, k^2\ f' + \beta\ k\ f + k\, f^{\,2} = 0. \tag{41}$$

where the integrating constant is considered to be zero. Balancing f'' and f^2 in (41) results $N+2=2N$, and so $N=2$. This offers a truncated series as the following form

$$f(\xi)=a_0+a_1\,Q(\xi)+a_2\,Q^2(\xi)\,. \tag{42}$$

By substituting (42) and its' derivatives with (37) into (41) and equating the coefficient of each power of $Q(\xi)$ to zero. We derive a system of algebraic equations as follows

$$-ca_0+k\beta a_0+ka_0^2=0\ , \tag{43}$$

$$-ca_1+k\beta a_1+k^2\alpha(log\,(m))a_1+ck^2(log\,(m))^2a_1+2ka_0a_1=0, \tag{44}$$

$$-k^2\alpha(log\,(m))a_1-3ck^2(log\,(m))^2a_1+ka_1^2-ca_2+k\beta a_2+2k^2\alpha(log\,(m))a_2+ \tag{45}$$

$$4ck^2(log\,(m))^2a_2+2ka_0a_2=0, \tag{46}$$

$$2ck^2(log\,(m))^2a_1-2k^2\alpha(log\,(m))a_2-10ck^2(log\,(m))^2a_2+2ka_1a_2=0, \tag{47}$$

$$6ck^2(log\,(m))^2a_2+ka_2^2=0. \tag{48}$$

Solving the above system, yields.

Case 1.

$$\beta=\frac{c+6ck^2(log\,(m))^2}{k},\quad a_0=0,\quad \alpha=5c\ (log\,(m))^2,\quad a_1=12ck\ (log\,(m))^2, \tag{49}$$

$$a_2=-6ck\ (log\,(m))^2. \tag{50}$$

Hence, the solution is formed as:

$$u_1(x,t)=\frac{12ck\ (log\,(m))^2}{1+d\ m^{\xi}}-\frac{6ck\ (log\,(m))^2}{(1+d\ m^{\xi})^2}, \tag{51}$$

where $\xi=k\,x-c\,t-x_0$.

Case 2.

$$\beta=\frac{c+6ck^2\ (log\,(m))^2}{k},\quad a_0=0,\quad \alpha=-5c\ (log\,(m))^2,\quad a_1=0, \tag{52}$$

$$a_2=-6ck\ (log\,(m))^2. \tag{53}$$

Therefore, the solution is formed as:

$$u_2(x,t)=-\frac{6ck\ (log\,(m))^2}{(1+d\ m^{\xi})^2}, \tag{54}$$

where $\xi=k\,x-c\,t-x_0$.

Case 3.

$$\beta=\frac{c-ck^2\ (log\,(m))^2}{k},\ a_0=0,\quad \alpha=0,\quad a_1=6ck\ (log\,(m))^2, \tag{55}$$

$$a_2=-6ck(log\,(m))^2. \tag{56}$$

Hence, the solution is formed as:

$$u_3\,(x,t) = \frac{6ck(log\,(m))^2}{1+d\;m^{\xi}} - \frac{6ck(log\,(m))^2}{\left(1+d\;m^{\xi}\right)^2}, \tag{57}$$

where $\xi = k\,x - c\,t - x_0$.

Case 4.

$$a_0 = -6ck(log\,(m))^2,\;\; \beta = \frac{c+6ck^2(log\,(m))^2}{k},\;\; \alpha = 5c(log\,(m))^2,\quad a_1 = 12ck\;\;Log\,[m]^2, \tag{58}$$

$$a_2 = -6ck(log\,(m))^2. \tag{59}$$

Thus, the solution is formed as:

$$u_4\,(x,t) = -6ck(log\,(m))^2 + \frac{12ck(log\,(m))^2}{1+d\;m^{\xi}} - \frac{6ck(log\,(m))^2}{\left(1+d\;m^{\xi}\right)^2}, \tag{60}$$

where $\xi = k\,x - c\,t - x_0$.

Case 5.

$$a_0 = -ck(log\,(m))^2,\quad \beta = \frac{c+ck^2(log\,(m))^2}{k},\quad \alpha = 0,\quad a_1 = 6ck(log\,(m))^2, \tag{61}$$

$$a_2 = -6ck(log\,(m))^2. \tag{62}$$

Then, the solution is formed as:

$$u_5\,(x,t) = -ck(log\,(m))^2 + \frac{6ck(log\,(m))^2}{1+d\;m^{\xi}} - \frac{6ck(log\,(m))^2}{\left(1+d\;m^{\xi}\right)^2}, \tag{63}$$

where $\xi = k\,x - c\,t - x_0$.

Case 6.

$$a_0 = 6ck(log\,(m))^2,\quad \beta = \frac{c-6ck^2(log\,(m))^2}{k},\quad \alpha = -5c(log\,(m))^2,\quad a_1 = 0, \tag{64}$$

$$a_2 = -6ck(log\,(m))^2. \tag{65}$$

Hence, the solution is formed as:

$$u_6\,(x,t) = 6ck(log\,(m))^2 - \frac{6ck(log\,(m))^2}{\left(1+d\;m^{\xi}\right)^2}, \tag{66}$$

where $\xi = k\,x - c\,t - x_0$.

7 Graphical illustrations of the solutions

We depict in this section some graphical illustrations of the obtained solutions for the BBM-Burgers equation. To reveal the clear picture of the obtained solutions, both the two and three dimensional plots for the solutions are given.

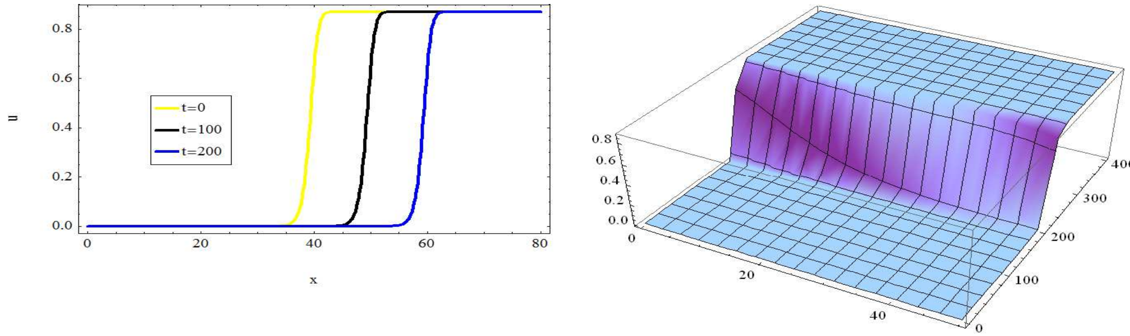

FIGURE 5. Graph of case (1) of the BBM-Burgers equation.

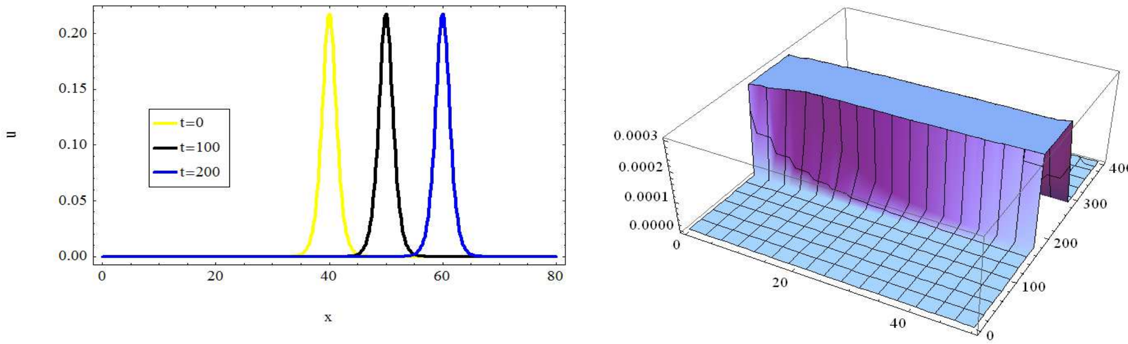

FIGURE 6. Graph of case (2) of the BBM-Burgers equation.

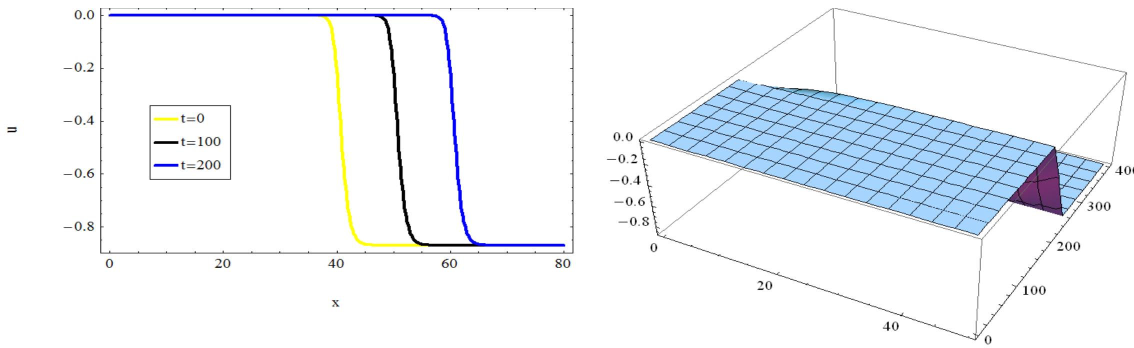

FIGURE 7. Graph of case (3) the BBM-Burgers equation.

8 Conclusion

In this paper, we have succeed two aims: Employing modified Kudryashov method for getting the exact solutions of the BBM-Burgers equation and the septic B-spline collocation finite element

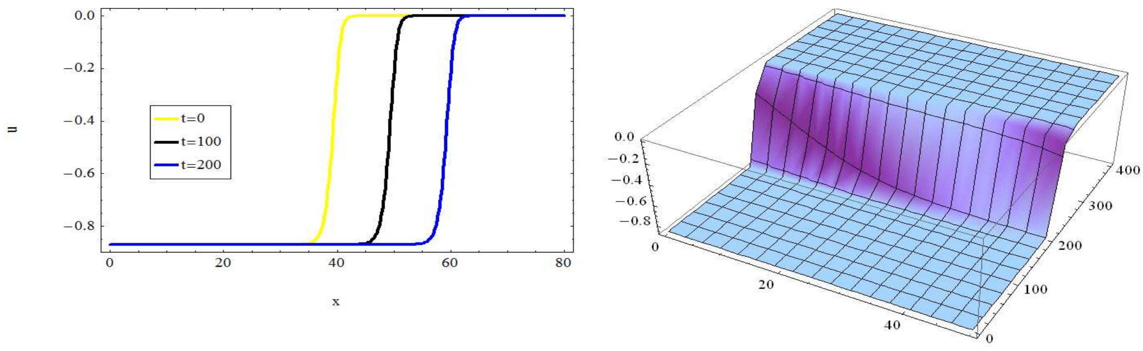

FIGURE 8. Graph of case (4) of the BBM-Burgers equation.

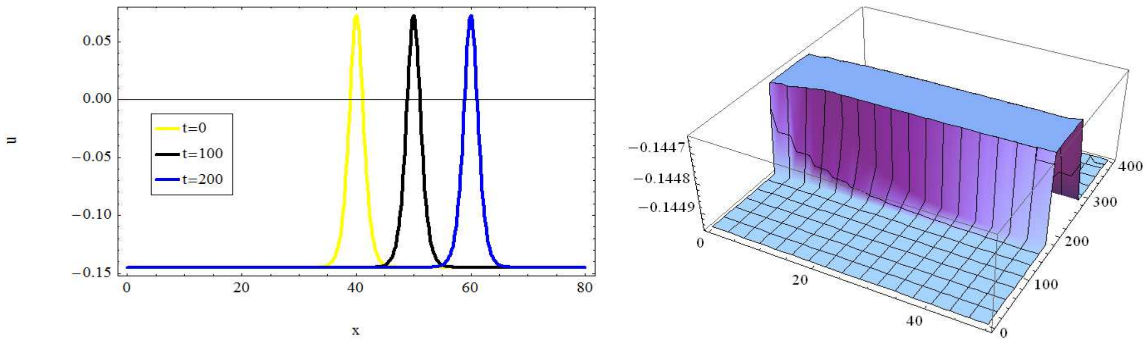

FIGURE 9. Graph of case (5) of the BBM-Burgers equation.

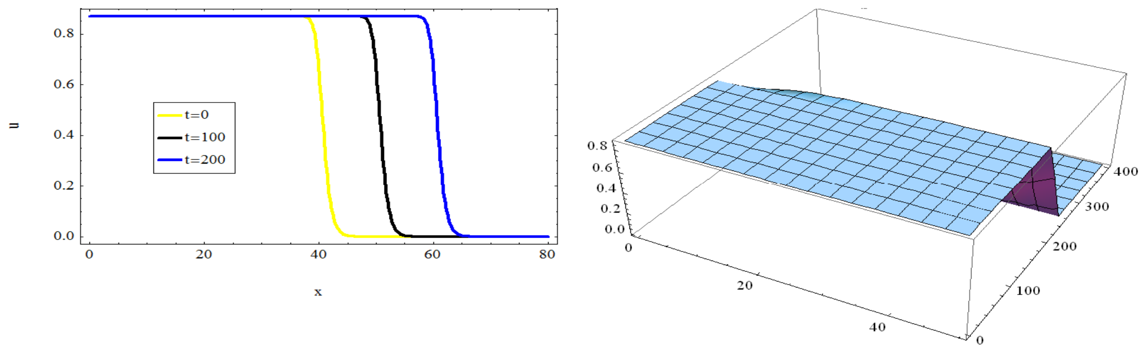

FIGURE 10. Graph of case (6) of the BBM-Burgers equation.

method for numerical study of travelling wave solutions of BBM-Burgers equation. The most marvelous part of the study is successful execution of both the schemes for getting both exact and numerical results. For the purpose of numerical experiments, we experimented our algorithm along with single solitary wave in which the exact solution is known. Stability analysis have been done and the linearized numerical scheme have been obtained unconditionally stable. The accuracy of the method is investigated both L_2 and L_∞ error norms and the invariant quantities I_1, I_2 and I_3. The obtained numerical results indicate that the error norms are satisfactorily small and the conservation laws are marginally constant in all computer program run. We can see that our numerical scheme for the equation is more accurate than the other earlier schemes found in the literature.

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