Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 7, No. 3, 2019, pp. 319-333

A numerical study using finite element method for generalized Rosenau-Kawahara-RLW equation

Seydi Battal Gazi Karakoc*
Department of Mathematics, Faculty of Science and Art,
Nevsehir Haci Bektas Veli University, 50300 Nevsehir, Turkey.
E-mail: sbgkarakoc@nevsehir.edu.tr

Samir Kumar Bhowmik
Department of Mathematics,
University of Dhaka, 1000 Dhaka, Bangladesh.
E-mail: bhowmiksk@gmail.com

Fuzheng Gao
School of Mathematics, Shandong University,
Shanda Nanlu 27, Jinan 250100, Shandong, P. R. China.
E-mail: fzgao@sdu.edu.cn

Abstract In this paper, we are going to obtain the soliton solution of the generalized Rosenau-Kawahara-RLW equation that describes the dynamics of shallow water waves in oceans and rivers. We confirm that our new algorithm is energy-preserved and unconditionally stable. In order to determine the performance of our numerical algorithm, we have computed the error norms L_2 and L_∞. Convergence of full discrete scheme is firstly studied. Numerical experiments are implemented to validate the energy conservation and effectiveness for longtime simulation. The obtained numerical results have been compared with a study in the literature for similar parameters. This comparison clearly shows that our results are much better than the other results.

Keywords. Generalized Rosenau-Kawahara-RLW equation, Finite element method, Collocation.

2010 Mathematics Subject Classification. 65N30, 65D07, 74S05,74J35, 76B25.

1. Introduction

The theory of nonlinear partial differential equations (PDEs) has made a significant development during the last decade. Especially we know that analyzing traveling-wave solutions for PDEs, using various methods, have long been an extensive interest for mathematicians, physicists, engineers and other scientists. At the same time, the existence of soliton kind solutions for PDEs is of special concern due to their common implementations in many physical structures for instance nonlinear optics, optical fibers, chemical physics, capillary-gravity, fluid dynamics and mechanics, plasmas, condensed matter, electro magnetics and any more. As known, a soliton is a specific kind of solitary waves which preserve its wave shape after inconformity with other

Received: 18 September 2018 ; Accepted: 18 November 2018.
* Corresponding author.

solitons. Some of the prevalently studied equations with soliton solutions are contain: Korteweg-de Vries (KdV) equation,

$$U_t + aUU_x + bU_{xxx} = 0, \tag{1.1}$$

was suggested propagation of long waves in shallow water waves [18, 27, 33, 34, 35, 48]; Regularized Long Wave (RLW) equation,

$$U_t + U_x + aUU_x - bU_{xxt} = 0, \tag{1.2}$$

was first introduced as a sample for small-amplitude long wave of water in a channel by Peregrine [6, 7, 8, 12, 14, 15, 16, 28, 31, 36, 42]; Kawahara equation

$$U_t + U_x + UU_x + U_{xxx} - U_{xxxxx} = 0, \tag{1.3}$$

came to exist in the theory of shallow water waves with surface tension [1, 9, 20, 24, 26, 41]; Rosenau equation

$$U_t + U_x + U_{xxxxt} + UU_x = 0, \tag{1.4}$$

and many others. In the research of the dynamics of intensive discrete systems, the status of wave-wave and wave-wall interactions can not be described using the well-recognized KdV equation. Moreover, the slope and the action of high amplitude waves are not be well estimated by the Eq. (1.1), since it was sampled under the assumption of weak anharmonicity [29]. To cope with the imperfection of the Eq. (1.1), Eq. (1.4) was proposed by Rosenau [38, 39]. If the viscous term U_{xxx} is added in Eq. (1.4), the newly equation is usually named Rosenau-KdV equation [2, 3, 4, 17, 25, 37, 38, 39, 40],

$$U_t + U_x + U_{xxx} + U_{xxxxt} + UU_x = 0. \tag{1.5}$$

When the viscous term $-U_{xxt}$ is involved in the Eq. (1.4), the obtained equation is usually termed the following Rosenau-RLW equation [5, 29, 30, 46, 47],

$$U_t + U_x - U_{xxt} + U_{xxxxt} + UU_x = 0. \tag{1.6}$$

If the another viscous term $-U_{xxxxx}$ is involved in the Eq. (1.5), the newly obtained equation is usually called the following Rosenau-Kawahara equation [10, 23, 50],

$$U_t + U_x + U_{xxx} + U_{xxxxt} - U_{xxxxx} + UU_x = 0. \tag{1.7}$$

The aim of this work is to search for the soliton solutions of the following generalized Rosenau-Kawahara-RLW equation

$$U_t + aU_x + bU^pU_x + cU_{xxx} - \alpha U_{xxt} + \lambda U_{xxxxt} - \zeta U_{xxxxx} = 0 \tag{1.8}$$

with the homogeneous boundary conditions

$$\begin{aligned} U(a,t) &= 0, & U(b,t) &= 0, \\ U_x(a,t) &= 0, & U_x(b,t) &= 0, \\ U_{xx}(a,t) &= 0, & U_{xx}(b,t) &= 0, \quad t > 0 \end{aligned} \tag{1.9}$$

and an initial condition

$$U(x,0) = U_0(x) \ a \le x \le b, \tag{1.10}$$

where p is a positive integer, α, λ are positive constants and a, b, c, ζ are all real constants. Generalized Rosenau-Kawahara-RLW equation has a limited number of numerical studies in the literature. Zuo [49] have implemented sech ansatz and tanh ansatz method to produce exact bright and dark 1-soliton solutions of the general Rosenau-Kawahara-RLW equation. Hea and Pan [21] have developed a three-level linearly implicit finite difference scheme based on sine-cosine method for solving the equation. Hea [22] generates the exact solitary wave solution for the perturbed Rosenau–Kawahara-RLW equation with power law nonlinearity and then improves a three-level linearly implicit difference algorithm for solving the equation. A three-level conservative fourth-order finite difference scheme for the initial boundary value problem of the generalized Rosenau-Kawahara-RLW equation has been developed by Wang and Dai [45].

In this work, the dynamics of shallow water waves that are modeled by generalized Rosenau–Kawahara-RLW equation along with septic B-spline collocation method will be studied. The rest of the paper can be outlined concisely as follows: In the second section, collocation finite element method has been applied to generalized Rosenau–Kawahara-RLW equation. The resulting system can be efficiently solved with a sort of the Thomas algorithm. Section 3 contains a linear stability analysis of the algorithm followed by section 4 which contains convergence of full discrete scheme. In the fifth section, motion of single soliton has examined. The obtained numerical results are given both in tabular and graphical format and the computed results are also compared with the studies available in the literature. Finally, the main findings of the paper are briefly summarized in a concluding section.

2. Collocation Method with Septic B-splines

For our numerical computations, solution area of the problem is limited over an interval $a \le x \le b$. Let the partition of the space interval $[a, b]$ into equally sized finite elements of length h at the points x_m like that $a = x_0 < x_1 < ... < x_N = b$ and $h = \frac{b-a}{N}$. The set of septic B-spline functions $\{\phi_{-3}(x), \phi_{-2}(x), \ldots, \phi_{N+3}(x)\}$ form a basis over the solution domain $[a, b]$. The numerical solution $U_N(x,t)$ is expressed in terms of the septic B-splines as

$$U_N(x,t) = \sum_{m=-3}^{N+3} \phi_m(x)\delta_m(t). \tag{2.1}$$

where $\delta_m(t)$ are time dependent parameters. Septic B-splines $\phi_m(x)$, $(m = -3, -2, ..., N+3)$ at the knots x_m are designated over the interval $[a, b]$ by Prenter [32]

$$\phi_m(x) = \frac{1}{h^7} \times$$
$$\begin{cases} (x - x_{m-4})^7, & [x_{m-4}, x_{m-3}], \\ (x - x_{m-4})^7 - 8(x - x_{m-3})^7, & [x_{m-3}, x_{m-2}], \\ (x - x_{m-4})^7 - 8(x - x_{m-3})^7 + 28(x - x_{m-2})^7, & [x_{m-2}, x_{m-1}], \\ (x - x_{m-4})^7 - 8(x - x_{m-3})^7 + 28(x - x_{m-2})^7 - 56(x - x_{m-1})^7, & [x_{m-1}, x_m], \\ (x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 + 28(x_{m+2} - x)^7 - 56(x_{m+1} - x)^7, & [x_m, x_{m+1}], \\ (x_{m+4} - x)^7 - 8(x_{m+3} - x)^7 + 28(x_{m+2} - x)^7, & [x_{m+1}, x_{m+2}], \\ (x_{m+4} - x)^7 - 8(x_{m+3} - x)^7, & [x_{m+2}, x_{m+3}], \\ (x_{m+4} - x)^7, & [x_{m+3}, x_{m+4}], \\ 0, & otherwise. \end{cases} \tag{2.2}$$

A characteristic finite interval $[x_m, x_{m+1}]$ is matched to the interval $[0, 1]$ by a local coordinate transformation defined by $h\xi = x - x_m$, $0 \leq \xi \leq 1$. Therefore septic B-splines (2.2) in terms of ξ over $[0, 1]$ are given as follows:

$$\begin{aligned} \phi_{m-3} &= 1 - 7\xi + 21\xi^2 - 35\xi^3 + 35\xi^4 - 21\xi^5 + 7\xi^6 - \xi^7, \\ \phi_{m-2} &= 120 - 392\xi + 504\xi^2 - 280\xi^3 + 84\xi^5 - 42\xi^6 + 7\xi^7, \\ \phi_{m-1} &= 1191 - 1715\xi + 315\xi^2 + 665\xi^3 - 315\xi^4 - 105\xi^5 + 105\xi^6 - 21\xi^7, \\ \phi_m &= 2416 - 1680\xi + 560\xi^4 - 140\xi^6 + 35\xi^7, \\ \phi_{m+1} &= 1191 + 1715\xi + 315\xi^2 - 665\xi^3 - 315\xi^4 + 105\xi^5 + 105\xi^6 - 35\xi^7, \\ \phi_{m+2} &= 120 + 392\xi + 504\xi^2 + 280\xi^3 - 84\xi^5 - 42\xi^6 + 21\xi^7, \\ \phi_{m+3} &= 1 + 7\xi + 21\xi^2 + 35\xi^3 + 35\xi^4 + 21\xi^5 + 7\xi^6 - \xi^7, \\ \phi_{m+4} &= \xi^7. \end{aligned} \tag{2.3}$$

For the problem, the finite elements are described with the interval $[x_m, x_{m+1}]$. Using Eq. (2.2) and Eq. (2.1), the nodal values of U_m, U'_m, U''_m, U'''_m and U^{iv}_m are given in concern with the element parameters δ_m by

$$\begin{aligned} &U_N(x_m, t) = \\ &U_m = \delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3}, \\ &U'_m = \tfrac{7}{h}(-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1} + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3}), \\ &U''_m = \tfrac{42}{h^2}(\delta_{m-3} + 24\delta_{m-2} + 15\delta_{m-1} - 80\delta_m + 15\delta_{m+1} + 24\delta_{m+2} + \delta_{m+3}), \\ &U'''_m = \tfrac{210}{h^3}(-\delta_{m-3} - 8\delta_{m-2} + 19\delta_{m-1} - 19\delta_{m+1} + 8\delta_{m+2} + \delta_{m+3}), \\ &U^{iv}_m = \tfrac{840}{h^4}(\delta_{m-3} - 9\delta_{m-1} + 16\delta_m - 9\delta_{m+1} + \delta_{m+3}) \end{aligned} \tag{2.4}$$

and the variation of U over the element $[x_m, x_{m+1}]$ is given by

$$U = \sum_{m=-3}^{N+3} \phi_m \delta_m. \tag{2.5}$$

When we define the collocation dots with the knots and use Eqs. (2.4) to evaluate U_m, its space derivatives and substitute into Eq. (1.8), this brings to a set of ordinary differential equations of the form

$$\begin{aligned}
&\dot{\delta}_{m-3} + 120\dot{\delta}_{m-2} + 1191\dot{\delta}_{m-1} + 2416\dot{\delta}_m + 1191\dot{\delta}_{m+1} + 120\dot{\delta}_{m+2} + \dot{\delta}_{m+3}\\
&-\tfrac{42}{h^2}\alpha(\dot{\delta}_{m-3} + 24\dot{\delta}_{m-2} + 15\dot{\delta}_{m-1} - 80\dot{\delta}_m + 15\dot{\delta}_{m+1} + 24\dot{\delta}_{m+2} + \dot{\delta}_{m+3})\\
&+\tfrac{840}{h^4}\lambda(\dot{\delta}_{m-3} - 9\dot{\delta}_{m-1} + 16\dot{\delta}_m - 9\dot{\delta}_{m+1} + \dot{\delta}_{m+3})\\
&+(a + bZ_m)\tfrac{7}{h}(-\delta_{m-3} - 56\delta_{m-2} - 245\delta_{m-1} + 245\delta_{m+1} + 56\delta_{m+2} + \delta_{m+3})\\
&+\tfrac{210}{h^3}c(-\delta_{m-3} - 8\delta_{m-2} + 19\delta_{m-1} + 19\delta_{m+1} + 8\delta_{m+2} + \delta_{m+3})+\\
&+\tfrac{2520}{h^5}\zeta(-\delta_{m-3} + 4\delta_{m-2} - 5\delta_{m-1} + 5\delta_{m+1} - 4\delta_{m+2} + \delta_{m+3}) = 0,
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
&Z_m = U_m\\
&= (\delta_{m-3} + 120\delta_{m-2} + 1191\delta_{m-1} + 2416\delta_m + 1191\delta_{m+1} + 120\delta_{m+2} + \delta_{m+3})^p.
\end{aligned}$$

If δ_i and its derivatives $\dot{\delta}_i$ in Eq. (2.6) are decoupled by Crank-Nicolson formula

$$\delta_i = \frac{\delta_i^{n+1} + \delta_i^n}{2}, \tag{2.7}$$

and usual finite difference approximation

$$\dot{\delta}_i = \frac{\delta_i^{n+1} - \delta_i^n}{\Delta t}, \tag{2.8}$$

we derive a repetition relationship between two time levels n and $n+1$ relating two unknown parameters δ_i^{n+1}, δ_i^n for $i = m-3, m-2, ..., m+2, m+3$

$$\begin{aligned}
&\gamma_1\delta_{m-3}^{n+1} + \gamma_2\delta_{m-2}^{n+1} + \gamma_3\delta_{m-1}^{n+1} + \gamma_4\delta_m^{n+1} + \gamma_5\delta_{m+1}^{n+1} + \gamma_6\delta_{m+2}^{n+1} + \gamma_7\delta_{m+3}^{n+1}\\
&= \gamma_7\delta_{m-3}^n + \gamma_6\delta_{m-2}^n + \gamma_5\delta_{m-1}^n + \gamma_4\delta_m^n + \gamma_3\delta_{m+1}^n + \gamma_2\delta_{m+2}^n + \gamma_1\delta_{m+3}^n,
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
&\gamma_1 = [1 - E + M - K(a + bZ_m) - L - T],\\
&\gamma_2 = [120 - 24E - 56K(a + bZ_m) - 8L + 4T],\\
&\gamma_3 = [1191 - 15E - 9M - 245(a + bZ_m) + 19L - 5T],\\
&\gamma_4 = [2416 + 80E + 16M],\\
&\gamma_5 = [1191 - 15E - 9M + 245(a + bZ_m) - 19L + 5T],\\
&\gamma_6 = [120 - 24E + 56K(a + bZ_m) + 8L - 4T],\\
&\gamma_7 = [1 - E + M + K(a + bZ_m) + L + T],\\
&m = 0, 1, \dots, N, E = \tfrac{42}{h^2}\alpha, M = \tfrac{840}{h^4}\lambda, K = \tfrac{7\Delta t}{2h}, L = \tfrac{210\Delta t}{2h^3}c, T = \tfrac{2520\Delta t}{2h^5}.
\end{aligned} \tag{2.10}$$

The system (2.9) involves of $(N+1)$ linear equations containing $(N+7)$ unknown coefficients $(\delta_{-3}, \delta_{-2}, \delta_{-1}, \ldots, \delta_{N+1}, \delta_{N+2}, \delta_{N+3})^T$. We need six additional restraints to obtain a unique solution for this system. These are obtained from the boundary conditions (1.9) and can be used to remove $\delta_{-3}, \delta_{-2}, \delta_{-1}$ and $\delta_{N+1}, \delta_{N+2}, \delta_{N+3}$ from the systems (2.9) which occurs a matrix equation for the $N+1$ unknowns $d^n = (\delta_0, \delta_1, \ldots, \delta_N)^T$ of the form

$$A\mathbf{d^{n+1}} = B\mathbf{d^n}. \tag{2.11}$$

Two or three inner iterations are implemented to the term $\delta^{n*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ at each time step to overcome the non-linearity caused by Z_m. Before the beginning of the solution procedure, initial parameters d^0 are established by using the initial condition and following derivatives at the boundaries;

$$\begin{array}{rcll}
U_N(x,0) & = & U(x_m,0); & m = 0,1,2,...,N \\
(U_N)_x(a,0) & = & 0, & (U_N)_x(b,0) = 0, \\
(U_N)_{xx}(a,0) & = & 0, & (U_N)_{xx}(b,0) = 0, \\
(U_N)_{xxx}(a,0) & = & 0, & (U_N)_{xxx}(b,0) = 0.
\end{array}$$

So we obtain the following matrix form for the initial vector d^0;

$$Vd^0 = w,$$

where

$$V = \begin{bmatrix}
1536 & 2712 & 768 & 24 & & & & & & \\
\frac{82731}{81} & \frac{210568.5}{81} & \frac{104796}{81} & \frac{10063.5}{81} & 1 & & & & & \\
\frac{9600}{81} & \frac{96597}{81} & \frac{195768}{81} & \frac{96474}{81} & 120 & 1 & & & & \\
 & & \ddots & & & & & & & \\
 & & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\
 & & & & 1 & 120 & \frac{96474}{81} & \frac{195768}{81} & \frac{96597}{81} & \frac{9600}{81} \\
 & & & & & 1 & \frac{10063.5}{81} & \frac{104796}{81} & \frac{210568.5}{81} & \frac{82731}{81} \\
 & & & & & & 24 & 768 & 2712 & 1536
\end{bmatrix}$$

$$d^0 = (\delta_0, \delta_1, \delta_2, ..., \delta_{N-2}, \delta_{N-1}, \delta_N)^T,$$

and

$$w = (U(x_0,0), U(x_1,0), ..., U(x_{N-1},0), U(x_N,0))^T.$$

3. Stability of the Algorithm

To investigate the stability analysis of the presented algorithm, it is suitable to use Von-Neumann theory. Presuming that the quantity U^p in the nonlinear term U^pU_x is locally constant. Substituting the Fourier mode $\delta_m^n = \xi^n e^{i\sigma mh}$, $(i = \sqrt{-1})$ into the form of (2.9) we obtain,

$$\begin{aligned}
&\xi^{n+1}(\eta_1 e^{i(m-3)\theta} + \eta_2 e^{i(m-2)\theta} + \eta_3 e^{i(m-1)\theta} + \eta_4 e^{im\theta} + \eta_5 e^{i(m+1)\theta} + \eta_6 e^{i(m+2)\theta} + \eta_7 e^{i(m+3)\theta}) \\
&= \xi^n(\eta_7 e^{i(m-3)\theta} + \eta_6 e^{i(m-2)\theta} + \eta_5 e^{i(m-1)\theta} + \eta_4 e^{im\theta} + \eta_3 e^{i(m+1)\theta} + \eta_2 e^{i(m+2)\theta} + \eta_1 e^{i(m+3)\theta})
\end{aligned} \tag{3.1}$$

where σ is mode number, h is the element size, $\theta = \sigma h$

$$\begin{aligned}
\eta_1 &= 1 - E + M - K(a + bZ_m) - L - T,\\
\eta_2 &= 120 - 24E - 56K(a + bZ_m) - 8L + 4T,\\
\eta_3 &= 1191 - 15E - 9M - 245(a + bZ_m) + 19L - 5T,\\
\eta_4 &= 2416 + 80E + 16M,\\
\eta_5 &= 1191 - 15E - 9M + 245(a + bZ_m) - 19L + 5T,\\
\eta_6 &= 120 - 24E + 56K(a + bZ_m) + 8L - 4T,\\
\eta_7 &= 1 - E + M + K(a + bZ_m) + L + T, \qquad\qquad m = 0, 1, \dots, N.
\end{aligned}$$

If we simplify the Eq. (3.1),

$$\xi = \frac{A + iB}{A - iB}$$

is obtained where

$$\begin{aligned}
A = {} & (2382 - 30E - 18M)\cos(\theta) + (240 - 48E)\cos(2\theta) + (2 - 2E + 2M)\cos(3\theta) +\\
& (2416 + 80E + 16M)\\
B = {} & (490K(1 + Z_m) - 38L - 10T)\sin(\theta) + (112K(1 + Z_m) + 16L - 8T)\sin(2\theta) +\\
& (2K(1 + Z_m) + 2L + 2T)\sin(3\theta).
\end{aligned} \tag{3.2}$$

According to Fourier stability analysis, for the given scheme to be stable, the condition $|\xi| < 1$ must be satisfied. Using a symbolic programming software or using simple calculations, since $a^2 + b^2 = a^2 + (-b)^2$ it becomes evident that the modulus of $|\xi|$ is 1. Hence the linearized algorithm is unconditionally stable.

4. Convergence of the full discrete scheme

In this article, we approximate a non-linear partial differential equation (1.8) using a higher order B-splines collocation approach at space predefined points for the space variable and a simple one step scheme for time. Efficiency of a computational algorithm depends on its implementation issues, high accuracy and numerical flexible stability conditions. Here we study the relevant concepts and key results without proof and cite sources of a more complete treatment. To be specific, we aim for a short discussion about the accuracy of the above mentioned space time scheme without a formal proof. One may consult [2, 8, 11, 43, 44] and the references therein for a detailed and settled theories. It is to note that we use some constants $C_i \geq 0$ here which not necessarily the same in all the cases. Usually global polynomial interpolations are used to integrate the solutions of differential equations for simple computational domain and when unknown curves are considered to be smooth enough. However, most engineering and physical problems are considered when the solutions are not sufficiently smooth to support global polynomial approximation and the computational domain is complicated. For these types of cases, finite element approximations play an important role and work very well to represent the solutions of the modelled problem.

Piecewise polynomial basis functions are smooth on each user defined sub-intervals which is one of the very important properties in approximation theory. It is very much helpful to analyze solutions approximated using such basis functions. On each spatial sub-intervals we have $p+1$ data values. Then there exists a polynomial of degree at most p passing through the data points. The error in such polynomial approximation is proportional to a power of the distance (usually denoted by h) between the data points [11, 43, 44]. As stated in the approximation above the scheme we used is based on spetic B-splines along with a spatial collocation approach. So the proposed scheme gives super-convergence at collocation points [19]. Also the scheme does not require an extra evaluation of an integral as of the Galerkin approach. So this approach is simpler and efficient to compute solutions.

Here by $H^k(\Omega)$ we mean the usual space of k times differentiable functions and by $\|\cdot\|_p$ we mean the standard $H^k(\Omega)$ norm. Here $\|\cdot\|_0$ stands for $L_2(\Omega)$ norm. Let v_h be an approximation to a function $v(x)\in H^k(\Omega)$ in Ω. Let h be the distance between the grids and $\Omega=\cup_i\Omega_i$, where $\Omega_i=[x_i,\ x_{i+1}]$, $x_{i+1}=x_i+h$. It is easy to observe [7, 13, 43, 44] that

$$\|v(x)-v_h(x)\|\leq Ch^{p+1}\|v\|_{p+1},$$

where $1\leq p<k$ and v_h stands for interpolation by piecewise-polynomials of degree r (considering $\Omega=\cup_i\Omega_i$). This error is preserved by Galerkin finite element approximation as well [2, 44]. Thus it is well established that

$$\|w(x)-w_h(x)\|\leq Ch^{m+1}\|w\|_{m+1},$$

where $1\leq m<p$ and if w_h is a suitable $B-$splines defined by a polynomial of degree less or equal k for any $w\in H_p(\Omega)$ [2, 11, 13]. In our present spatial approximation we apply septic B-splines. Thus it is evident that theoretically $\mathcal{O}(h^8)$ accuracy has been achieved from such a spatial approximation in $L_2(\Omega)$ [2]. Here a forward difference scheme is accurate of $\mathcal{O}(\Delta t)$ in $L_2([0\ T])$ norm for some $T>0$ [44]. So for the full discrete scheme the accuracy can be presented by the following inequality

$$\|u(x,t)-u_h(x,t)\|\leq C_1h^8+C_2\Delta t,$$

for a suitable $C_1\geq 0$ and $C_2\geq 0$.

5. Numerical Results and Discussion

In this section, in order to confirm the correction of our algorithm, some numerical test problems will calculate. For the numerical simulations of the motion of single solitary wave for which exact solutions have known before, two sets of parameters are used and probed. The initial value problem (1.8) − (1.10) has following conservative quantity;

$$I_E=\frac{1}{2}\int_a^b[U^2(x,t)+\alpha U_x^2(x,t)+\lambda U_{xx}^2(x,t)]dx \tag{5.1}$$

which corresponds to the energy of the shallow water waves [22]. The accuracy and efficiency of the numerical algorithm is controlled by both the error norm [2]

$$L_2=\left\|U^{exact}-U_N\right\|_2\simeq\sqrt{h\sum_{J=0}^{N}\left|U_j^{exact}-(U_N)_j\right|^2},$$

and the error norm

$$L_\infty = \left\|U^{exact} - U_N\right\|_\infty \simeq \max_j \left|U_j^{exact} - (U_N)_j\right|.$$

5.1. **Case 1.** For the first numerical calculation, we choose the parameters $p = 2$, $\Delta t = 0.005$ with various values of h over the interval $[-40, 200]$. For this case, since exact solution of the problem is [50]

$$U(x,t) = \frac{3}{4}\frac{\sqrt{370} - 5\sqrt{10}}{\sqrt{5\sqrt{37} - 29}} \sec h^2[\frac{\sqrt{\sqrt{37} - 5}}{4}(x - \frac{33 - 5\sqrt{37}}{5\sqrt{37} - 29})t], \tag{5.2}$$

the initial condition for the problem is taken as

$$U(x,0) = \frac{3}{4}\frac{\sqrt{370} - 5\sqrt{10}}{\sqrt{5\sqrt{37} - 29}} \sec h^2[\frac{\sqrt{\sqrt{37} - 5}}{4}x]. \tag{5.3}$$

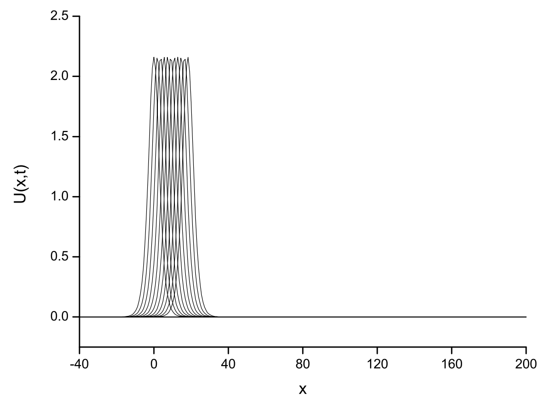
The values of invariant and error norms for single solitary wave over the interval

Table 1. Invariant and error norms for single soliton for $p = 2$ over the interval $-40 \leq x \leq 200$.

$h = 1.6$	I_E	$L_2 \times 10^3$	$L_\infty \times 10^3$
$t = 1$	25.4521536	1.91446	0.94469
$t = 5$	25.4515106	2.19677	0.94948
$t = 10$	25.4513649	3.75806	1.30782
$h = 1.0$			
$t = 1$	25.4517802	0.09357	0.04842
$t = 5$	25.4517802	0.13110	0.05753
$t = 10$	25.4517802	0.16745	0.07394
$h = 0.8$			
$t = 1$	25.4517766	0.04106	0.02205
$t = 5$	25.4517766	0.05276	0.02288
$t = 10$	25.4517766	0.06242	0.02787
[21]	25.4514057	266	103.2
[45]	25.4474394	0.3279	0.1177
$h = 0.6$			
$t = 1$	25.4517763	0.01333	0.00687
$t = 5$	25.4517763	0.01460	0.00579
$t = 10$	25.4517763	0.01529	0.00604

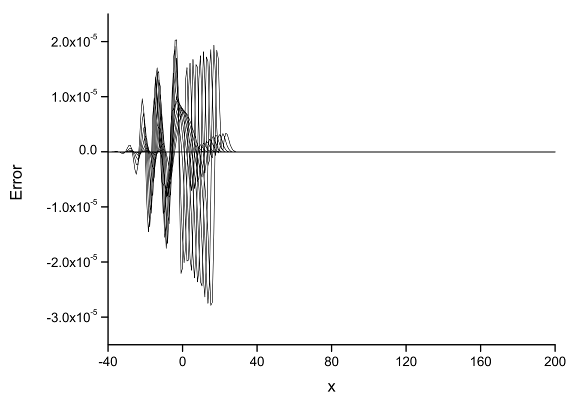
$-40 \leq x \leq 200$ are tabulated in Table 1. From Table 1, it is seen that the error norms obtained by our method are found much better than the others for $h = 0.8$ and the invariant I_E changes from its initial values by less than 7.9×10^{-4} for $h = 1.6$ whereas the changes of invariant approach to zero for $h = 1.0, 0.8$ and 0.6. Also, we have detected that the quantities of the error norms L_2 and L_∞ are sufficiently small during the computer run. Thus, we can say our method is marginally conservative. The profiles of the single soliton for $h = 0.8$, $\Delta t = 0.005$ at times $t = 1, 2, 3, ..., 10$ are given in Figure 1. We observed from the Figure 1, single soliton travels to the right at a constant speed and keeps its amplitude and form with increasing time as not surprisingly. The amplitude is 2.1597256 at $t = 0$ and located at $x = 0.0$, while it is 2.1580500 at $t = 10$ and located at $x = 18.4$. The absolute difference in amplitudes at times $t = 0$ and $t = 10$ is 1.67×10^{-3} so that there is a little change between amplitudes. To demonstrate the errors between the exact and numerical results over

FIGURE 1. Motion of single soliton for $p = 2$, $h = 0.8$, $\Delta t = 0.005$ over the interval $[-40, 200]$ at time $t = 0, 1, 2, ..., 10$.



the solution domain, error distributions at time $t = 1, 2, ..., 10$ are depicted graphically in Figure 2. The maximum errors are between -3×10^{-5} - 2×10^{-5} and occur around the central position of the solitary wave.

FIGURE 2. Errors for $p = 2$, $h = 0.8$, $\Delta t = 0.005$ at $t = 1, 2, ..., 10$.



5.2. **Case 2.** For the second numerical calculation, we have used the parameters $p = 4$, $\Delta t = 0.005$ with different values of h over the interval $[-40, 240]$. For this case, we have found the analytical solution of the problem [21]

$$U(x,t) = (\frac{40(\sqrt{127}-10)^2}{3(10\sqrt{127}-109)})^{\frac{1}{4}} \sec h[\frac{\sqrt{\sqrt{127}-10}}{3}(x-\frac{118-10\sqrt{127}}{10\sqrt{127}-109})]t, \quad (5.4)$$

so the initial condition for this problem is taken as

$$U(x,0) = (\frac{40(\sqrt{127}-10)^2}{3(10\sqrt{127}-109)})^{\frac{1}{4}} \sec h[\frac{\sqrt{\sqrt{127}-10}}{3}x]. \quad (5.5)$$

The experiment is run from $t = 0$ to $t = 10$ and values of the invariant quantities and error norms are listed in Table 2. Table 2 shows that values of invariant are almost

TABLE 2. Invariant and error norms for single soliton for $p = 4$ over the interval $-40 \leq x \leq 240$.

$h = 1.6$	I_E	$L_2 \times 10^3$	$L_\infty \times 10^3$
$t = 1$	13.5736473	8.40940	4.09076
$t = 5$	13.5549680	17.44637	7.88485
$t = 10$	13.5725435	31.25806	1185995
$h = 1.0$			
$t = 1$	13.5658554	0.17182	0.09781
$t = 5$	13.5658605	0.25653	0.12195
$t = 10$	13.5658562	0.39970	0.18451
$h = 0.8$			
$t = 1$	13.5658560	0.07310	0.04803
$t = 5$	13.5658560	0.12211	0.05817
$t = 10$	13.5658560	0.20207	0.09192
[21]	13.5656656	154.3	58.39
[45]	13.5637615	1.684	0.7812
$h = 0.7$			
$t = 1$	13.5658553	0.04322	0.02816
$t = 5$	13.5658553	0.06912	0.03244
$t = 10$	13.5658553	0.11088	0.05002

constant as the time increases. It is noticeably seen from the table that the error norms obtained by our method are found much better than the others for $h = 0.8$ and the invariant I_E changes from its initial value by less than 1.10×10^{-3} for $h = 1.6$ and 8×10^{-7} for $h = 1.0$ whereas the changes of invariant approach to zero for $h = 0.8$ and 0.7. Also, we have found out error norms L_2 and L_∞ are obtained sufficiently small during the computer run. Therefore our method is sensibly conservative.
For visual representation, the simulations of single soliton for values of $h = 0.8$, $\Delta t = 0.005$ at times $t = 1, 2, 3, ..., 10$ are illustrated in Figure 3. It is understood from this figure that the numerical scheme performs the motion of propagation of a single solitary wave, which moves to the right at nearly unchanged speed and conserves its amplitude and shape with increasing time. The amplitude is 1.5529458 at $t = 0$ and located at $x = 0.0$, while it is 1.5528307 at $t = 10$ and located at $x = 14.4$. The absolute difference in amplitudes at times $t = 0$ and $t = 10$ is 1.15×10^{-4} so that there is a little change between amplitudes. Error distributions at time $t = 1, 2, ..., 10$ are shown graphically in Figure 4. As it is seen, the maximum errors are between $1\times\ 10^{-4}$ - $6\times\ 10^{-5}$ and occur around the central position of the solitary wave.

FIGURE 3. Motion of single soliton for $p = 4$, $h = 0.8$, $\Delta t = 0.005$ over the interval $[-40, 240]$ at time $t = 0, 1, 2, ..., 10$.

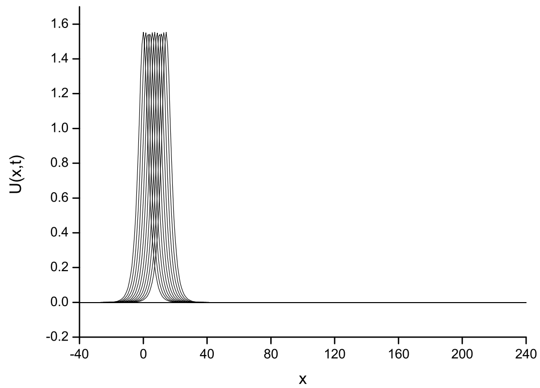
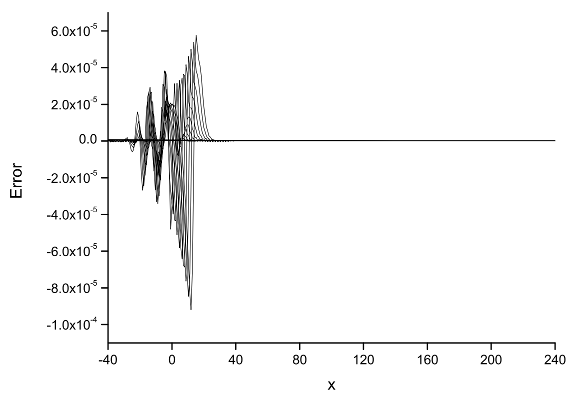


FIGURE 4. Errors for $p = 4$, $h = 0.8$, $\Delta t = 0.005$ at $t = 1, 2, ..., 10$.



6. Conclusion

In this paper, we have derived soliton solutions of the generalized Rosenau-Kawahara-RLW equation based on septic B-spline collocation finite element method. Numerical solutions of the 1-soliton are obtained for two cases. The newly suggested scheme produces very highly accurate results and the conserved quantities are almost constant during the simulation both $p = 2$ and $p = 4$. The main energy conservative property is preserved by the current numerical scheme. The method is shown to be unconditionally stable. Also convergence of full discrete scheme is generated. In summary, the derived method is highly accurate and appropriate for simulated solution of the generalized Rosenau-Kawahara-RLW equation. Therefore, we can say that the suggested method is also very efficient for solving nonlinear partial differential equations.

Acknowledgment

Seydi Battal Gazi Karakoc's research is supported by Nevsehir Haci Bektas Veli University Scientific Research Projects Coordination Unit (Turkey) no. BAP18F36.

References

[1] T. Ak and S. B. G. Karakoc, *A numerical technique based on collocation method for solving modified Kawahara equation,* Journal of Ocean Engineering and Science, *3* (2018), 67–75.

[2] T. Ak, S. Dhawan, S. B. G. Karakoc, S. K. Bhowmik, and K. R. Raslan, *Numerical study of Rosenau-KdV equation using finite element method based on collocation approach,* Mathematical Modelling and Analysis, *22*(3) (2017), 373–388.

[3] T. Ak, S. B. G. Karakoc, and A. Biswas, *Numerical scheme to dispersive shallow water waves,* Journal of Computational and Theoretical Nanoscience, *13*(10) (2016), 7084–7092.

[4] T. Ak, S. B. G. Karakoc, and H. Triki, *Numerical simulation for treatment of dispersive shallow water waves with Rosenau-KdV equation,* The European Physical Journal Plus, *13*(10) (2016), 1–15.

[5] N. Atouani and K. Omrani, *Galerkin finite element method for the Rosenau-RLW equation*, Computers and Mathematics with Applications, *66* (2013), 289–303.

[6] T. B. Benjamin, J. L. Bona, and J. J. Mahony, *Model equations for long waves in non-linear dispersive systems*, Phil. Trans. Roy. Soc. London A, *272* (1972), 47–78.

[7] S. K. Bhowmik, *Piecewise polynomial approximation of a nonlocal phase transitions model*, Mathematical Analysis and Applications, *420*(2) (2014), 1069–1094.

[8] S. K. Bhowmik, R. Belbaki, T. Z. Boulmezaoud, and S. Mziou, *Solving two dimensional second order elliptic equations in exterior domains using the inverted finite elements method*, Computers & Mathematics with Applications, *72*(9) (2014), 2315–2333.

[9] A. Biswas, *Soliton perturbation theory for the modified Kawahara equation,* Appl. Appl. Math., *3*(2) (2008), 218–223.

[10] A. Biswas, H. Triki, and M. Labidi, *Bright and dark solitons of the Rosenau-Kawahara equation with power law nonlinearity*, Phys. Wave. Phenom., *19* (2011), 24–29.

[11] P. B. Bochev and M. D. Gunzburger, *Least-squares finite element methods*, Springer, 2008.

[12] I. Dag, B. Saka, and D. Irk, *Galerkin method for the numerical solution of the RLW equation using quintic B-splines*, J. Comput. Appl. Math., *190* (2006), 532–547.

[13] S. Dhawan, S. K. Bhowmik, and S. Kumar, Galerkin-least square b-spline approach toward advection diffusion equation, Applied Mathematics and Computation, *261* (2015), 128–140.

[14] T. S. EL-Danaf, K. R. Raslan, and K. K. Ali, *New numerical treatment for the generalized regularized long wave equation based on finite difference scheme*, Int. J. of S. Comp. and Eng., *4* (2014), 16–24.

[15] T. S. EL-Danaf, K. R. Raslan, and K. K. Ali, *Collocation method with cubic B-splines for solving the GRLW equation*, Int. J. of Num. Meth. and Appl., *15*(1) (2016), 39–59.

[16] T. S. EL-Danaf, K. R. Raslan, and K. K. Ali, *Non-polynomial spline method for solving the generalized regularized long wave equation*, Communication in Mathematical Modeling and Applications, *2* (2017), 1–17.

[17] A. Esfahani, *Solitary wave solutions for generalized Rosenau-KdV equation*, Commun. Theor. Phys., *63* (2013), 1–7.

[18] C. S. Gardner, J. M. Green, M. D. Kruskal, and R. M. Miura, *Method for solving Korteweg- de Vries equation*, Phys. Rev., *19*(19) (1967), 1095–1097.

[19] H. Gomez and L. De Lorenzis, *The variational collocation method*, Computer Methods in Applied Mechanics and Engineering, *309* (2016), 152–181.

[20] M. Haragus, E. Lombardi, and A. Scheel, *Spectral stability of wave trains in the Kawahara equation,* J. math. fluid mech., *8* (2006), 482–509.

[21] D. Hea and K. Pan, *A linearly implicit conservative difference scheme for the generalized Rosenau–Kawahara-RLW equation*, Applied Mathematics and Computation, *271* (2015), 323–336.

[22] D. Hea, *Exact solitary solution and a three-level linearly implicit conservative finite difference method for the generalized Rosenau–Kawahara-RLW equation with generalized Novikov type perturbation*, Nonlinear Dynamics, *85*(1) (2016), 479–498.

[23] J. Hu, Y. Xu, B. Hu, and X. Xie, *Two conservative difference schemes for Rosenau-Kawahara equation*, Adv. Math. Phys., *2014* (2014), Article ID: 217393, 11 pages.

[24] S. B. G. Karakoc, H. Zeybek, and T. Ak, *Numerical solutions of the Kawahara equation by the septic B-spline collocation method,* Statistics, Optimization and Information Computing, *2*(3) (2014), 211–221.

[25] S. B. G. Karakoc, and T. Ak, *Numerical solution of Rosenau-KdV equation using subdomain finite element method,* New Trends in Mathematical Sciences, *4*(1) (2016), 223–235.

[26] T. Kawahara, *Oscillatory solitary waves in dispersive media*, J. Phys. Soc. Jpn., *33* (1972), 260–264.

[27] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary wave*, Philosophical 12 Magazine, *39* (1895), 422–443.

[28] K. Omrani, *The convergence of the fully discrete Galerkin approximations for the Benjamin–Bona–Mahony (BBM) equation*, Appl. Math. Comput., *180* (2006), 614–621.

[29] X. Pan and L. Zhang, *Numerical simulation for general Rosenau-RLW Equation: An Average Linearized Conservative Scheme*, Mathematical Problems in Engineering, *2012* (2012), Article ID: 517818, 15 pages.

[30] X. Pan, K. Zheng, and L. Zhang, *Finite difference discretization of the Rosenau-RLW equation*, Appl. Anal., *92* (2013), 2578–2589.

[31] D. H. Peregrine, *Calculations of the development of an undular bore*, J. Fluid Mech., *25* (1966), 321–330.

[32] P. M. Prenter, *Splines and variational methods*, John Wiley & Sons, New York, NY.USA, 1975.

[33] K. R. Raslan, T. S. El-Danaf, and K. K. Ali, *Application of linear combination between cubic B-spline collocation methods with different basis for solving the KdV equation*, Computational Methods for Differential Equations, *4*(3) (2016), 191–204.

[34] K. R. Raslan, T. S. El-Danaf, and K. K. Ali, *Collocation method with Quantic b-spline method for solving hirota-satsuma coupled KDV equation*, International Journal of Applied Mathematical Research, *5*(2) (2016), 123–131.

[35] K. R. Raslan, T. S. El-Danaf, and K. K. Ali, *New numerical treatment for solving the KDV equation*, Journal of Abstract and Computational Mathematics, 2(1) (2017), 1-12.

[36] K. R. Raslan, K. K. Ali, and M. A. Shallal, *Solving the space-time fractional RLW and MRLW equations using modified extended tanh method with the Riccati equation*, British Journal of Mathematics and Computer Science, *21*(4) (2017), 1–15.

[37] P. Razborova, H. Triki, and A. Biswas, *Perturbation of disperive shallow water waves*, Ocean Eng., *63* (2013), 1–7.

[38] P. Rosenau, *A quasi-continuous description of a nonlinear transmission line*, Phys. Scripta., *34* (1986), 827–829 .

[39] P. Rosenau, *Dynamics of dense discrete systems*, Progr. Theory. Phys., *79* (1988), 1028–1042.

[40] A. Saha, *Topological 1-soliton solutions for the generalized Rosenau-KdV equation*, Fund. J. Math. Phys., *2*(1) (2012), 19–25.

[41] T. Sirendaoreji, *New exact travelling wave solutions for the Kawahara and modified Kawahara equations*, Chaos Solitons Fractals, *19* (2004), 147–150.

[42] A. A. Soliman and M. H. Hussien, *Collocation solution for RLW equation with septic spline*, Appl. Math. Comput., *161* (2005), 623–636.

[43] A. Suli and D. Mayers, *An introduction to numerical analysis*, Cambridge University Press, 2003.

[44] V. Thomee, *Galerkin finite element methods for parabolic problems*, Second edition, Springer, 2006.

[45] X. Wang and W. Dai, *A new implicit energy conservative difference scheme with fourth-order accuracy for the generalized Rosenau-Kawahara-RLW equation*, Computational and Applied Mathematics, (2018), 1-22. DOI 10.1007/s40314-018-0685-4.

[46] B. Wongsaijai and K. Poochinapan, *A three-level average implicit nite difference scheme to solve equation obtained by coupling the Rosenau-KdV equation and the Rosenau-RLWequation*, Applied Mathematics and Computation, *245* (2014), 289–304.

[47] N. M. Yagmurlu, B. Karaagac, and S. Kutluay, *Numerical solutions of Rosenau-RLW equation using Galerkin cubic B-spline finite element method*, American Journal of Computational and Applied Mathematics, *7*(1) (2017), 1-10.

[48] N. J. Zabusky and M. D. Kruskal, *Interaction of solitons in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett., *15*(6) (1965), 240–243.

[49] J. M. Zuo, *Soliton solutions of a general Rosenau-Kawahara-RLW equation*, Journal of Mathematics Research, *7*(2) (2015), 24–28.

[50] J. M. Zuo, *Solitons and periodic solutions for the Rosenau-KdV and Rosenau-Kawahara equations*, Appl. Math. Comput., *215*(2) (2009), 835–840.