A C^0 Finite Element Method for the Biharmonic Problem without Extrinsic Penalization

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A symmetric C^0 finite element method for the biharmonic problem is constructed and analyzed. In our approach, we introduce one-sided discrete second-order derivatives and Hessian matrices to formulate our scheme. We show that the method is stable and converge with optimal order in a variety of norms. A distinctive feature of the method is that the results hold without extrinsic penalization of the gradient across interelement boundaries. Numerical experiments are given that support the theoretical results, and the extension to Kirchhoff plates is also discussed. Numer Methods Partial Differential Eq 30: 1254–1278, 2014

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I. INTRODUCTION

In this article, we consider a new symmetric C^0 finite element method for the biharmonic problem with clamped-plate boundary conditions:

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{1.1c}$$

Here, $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a open, connected, polyhedral domain, $f \in H^{-1}(\Omega)$ is a given forcing function, $\Delta^2 := \sum_{i,j=1}^{d} \partial^4/\partial x_i^2 \partial x_j^2$ denotes the biharmonic operator, and $\partial u/\partial n := \nabla u \cdot \boldsymbol{n}$,

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where $\boldsymbol{n}$ denotes the outward unit normal of the boundary $\partial\Omega$. A function $u \in H_0^2(\Omega)$ is defined to be a solution to (1.1) provided

$$\int_\Omega D^2u : D^2v dx = \int_\Omega f v dx \quad \forall v \in H_0^2(\Omega), \tag{1.2}$$

where $D^2u : D^2v := \sum_{i,j=1}^{d} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}$ denotes the Frobenius inner product between the two Hessian matrices D^2u and D^2v. Additional notation is given in the subsequent section.

Due to their simplicity, computational efficiency, and availability on commercial software, C^0 finite element methods are an attractive choice to compute fourth-order elliptic problems. The first such method was introduced in [1], where discontinuous Galerkin techniques were utilized to construct an interior penalty (IP) C^0 method. This method was subsequently analyzed in considerable detail in two dimensions (2D) on polygonal domains in [2, 3]. A defining feature of this method is the presence of a user-defined penalization parameter which must be taken sufficiently large to ensure stability and convergence of the scheme. In general, it is not known *a priori* neither how large to take the penalization term nor is it known what the optimal value (with respect to approximation, conditioning, etc.) should be. In contrast, the weakly overpenalized IP method given in [4] is stable for any positive penalization parameter. However, due to the inconsistent scaling of the method, the condition number is of order $\mathcal{O}(h^{-6})$ without preconditioning.

Recently, a new class of methods have been constructed for fourth-order problems based on a local-discontinuous Galerkin (LDG) approach [5, 6]. This class of methods is based on a mixed formulation of the fourth-order problem and the choice of appropriate numerical traces. Similar to the C^0 IP methods, these schemes include an extrinsic term that penalizes the jumps of the gradient across interelement boundaries. An advantage of these schemes is that they are stable for any positive penalization parameter while still retaining the usual $\mathcal{O}(h^{-4})$ conditioning.

The class of C^0 finite element methods constructed in this article has a similar structure of the LDG-type methods in [5, 6]. However, rather than defining our method through the use of numerical traces, we use one-sided discrete second-order derivatives to construct our scheme; see Definition 2.1. Using this approach, we show that the new method is stable and converge with optimal order in a variety of norms. A distinctive feature of the method is that the results hold without extrinsic penalization. Possible advantages of the scheme include computational simplicity as well as the lack of tuning of a penalty parameter to ensure the stability and convergence of the method. As far as we are aware, the proposed C^0 method is the first symmetric method with these features on a general class of triangulations.

This work is motivated by the papers [7, 8], where a discrete differential calculus framework for discontinuous functions is introduced. Here, one-sided discrete first-order derivatives are defined and various calculus identities (e.g., integration by parts and product rule) are established. Recently, the second author and Lewis used this discrete calculus framework to construct an LDG-type scheme for the Poisson problem that requires no penalization [8]. The natural generalization of this method for the biharmonic problem is presented here.

The organization of this article is as follows. In the next section, we provide the notation used throughout the article and define the one-sided discrete second-order operators and discrete Hessian matrices. With these definitions set, we define the C^0 method and compare the method to the local-continuous-discontinuous Galerkin method given in [5]. In Section III, we state the main results of the article, namely, existence, uniqueness, and optimal-order estimates in the energy norm and H^1 norm. The next two sections, the bulk of the article, is devoted to proving these results. In Section IV, we prove some preliminary identities and establish some results of the Morley finite element space. The proofs of the main results are then given in Section V. Finally, we discuss some extensions of the method to Kirchhoff plates in Section VI and provide some numerical experiments in Section VII.

II. THE FINITE ELEMENT METHOD

A. Notation

Let $\mathcal{T}_h$ be a conforming, quasiuniform triangulation of the domain Ω, and let ϵ_h denote the set of $(d-1)$-dimensional simplices in $\mathcal{T}_h$, for example, the set of faces ($d = 3$) or edges ($d = 2$) in $\mathcal{T}_h$. In addition, we set $\mathcal{E}_h^B$ to be the set of boundary $(d-1)$-dimensional simplices and set $\mathcal{E}_h^I := \mathcal{E}_h \backslash \mathcal{E}_h^B$, the set of interior $(d-1)$-dimensional simplices in $\mathcal{T}_h$. For a number $p \in [1, \infty]$ and $m \geq 0$, we define the piecewise Sobolev spaces with respect to the triangulation as $W^{m,p}(\mathcal{T}_h) := \prod_{T \in \mathcal{T}_h} W^{m,p}(T)$. For notational convenience, we set the special cases $H^m(\mathcal{T}_h) := W^{m,2}(\mathcal{T}_h), \mathbb{V}^h := H^2(\mathcal{T}_h)$, and define the piecewise Sobolev norms

$$|v|^2_{H^m(\mathcal{T}_h)} := \sum_{T \in \mathcal{T}_h} |v|^2_{H^m(T)}.$$

The piecewise L^2-inner product over the triangulation $\mathcal{T}_h$ is given by

$$(v, w)_{\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} \int_T v^\circ w dx,$$

and the L^2-inner product over a subset $\mathcal{S}_h \subset \mathcal{E}_h$ is given by

$$\langle v, w \rangle_{\mathcal{S}_h} := \sum_{e \in \mathcal{S}_h} \int_e v^\circ w ds,$$

where $\circ$ denotes the product, inner product, or Frobenius inner product depending on whether v,w are scalar, vector, or matrix-valued functions. If $v, w \in L^2(\Omega)$, then we simply write $(v, w) := (v, w)_{\mathcal{T}_h}$. We also denote by $\langle \cdot, \cdot \rangle$ the pairing between some Banach space and it's dual.

We denote by $\mathcal{P}_r(D)$ the space of polynomials with domain $D \subset \Omega$ and degree not exceeding $r (\geq 0)$. The space of piecewise polynomials with respect to the triangulation is given by $V_r^h := \prod_{T \in \mathcal{T}_h} \mathcal{P}_r(T)$. We also set $\mathcal{V}_r^h = V_r^h \cap H_0^1(\Omega)$, the globally continuous Lagrange finite element space of degree r. We note the obvious inclusions $\mathcal{V}_r^h \subset V_r^h \subset \mathbb{V}^h$. In addition, we set $\tilde{\mathbb{V}}^h := [\mathbb{V}^h]^{d \times d}$, $\tilde{\boldsymbol{V}}_r^h := [V_r^h]^{d \times d}$ and $\tilde{\mathcal{V}}_r^h := [\boldsymbol{V}_r^h]^{d \times d}$.

Let $T^\pm \in \mathcal{T}_h$ with $e = \partial T^+ \cap \partial T^- \in \mathcal{E}_h^I$. Without loss of generality, we assume that the global labeling number of T^+ is smaller than that of T^-. The unit normal of e is defined by $\boldsymbol{n}_e = (n_e^{(1)}, n_e^{(2)}, \ldots, n_e^{(d)})^t := \boldsymbol{n}_{T^+}|_e = -\boldsymbol{n}_{T^-}|_e$, and jumps and average of a function $v \in H^1(\mathcal{T}_h)$ are defined, respectively, by

$$[\![v]\!]|_e := (v^+ - v^-)|_e, \quad \{\{v\}\}|_e := \frac{1}{2}(v^+ + v^-)|_e.$$

Above, $v^\pm := v|_{T^\pm}$ is the restriction of v to the simplex $T^\pm$. On a boundary simplex $e \in \mathcal{E}_h^b$, we simply take $[\![v]\!]|_e = \{\{v\}\}|_e = v|_e$.

B. Discrete Derivatives

In this section, we state the definitions of the discrete second-order derivatives which are the building blocks of our scheme. The definition is motivated by the following elementary identity which holds for all $v \in H^2(\Omega)$ and $w \in V_r^h$.

$$\int_\Omega \frac{\partial^2 v}{\partial x_i \partial x_j} w dx = -\sum_{T\in\mathcal{T}_h} \int_T \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} dx + \sum_{e\in\mathcal{E}_h} \int_e \text{tr}\left(\frac{\partial v}{\partial x_i}\right) n_e^{(j)} [\![w]\!] ds, \tag{2.1}$$

where tr$(\cdot)$ denotes the usual trace operator. We would now like to consider a discrete operator from $\mathbb{V}^h$ to V_r^h which maps to the right-hand side of (2.1). However, the trace operator acting on $v \in \mathbb{V}^h$ is multivalued on interior edges, and therefore (2.1) does not apply for such functions. Instead, we shall consider the following three trace operators acting on the space $\mathbb{V}^h$ (see [7] for more details).

Definition 2.1. *Let $e \in \mathcal{E}_h^I$ with $e = \partial T^+ \cap \partial T^-$. The trace operators $\mathcal{Q}_j^\pm, \mathcal{Q}_j$ on e in the direction x_j are defined as*

$$\mathcal{Q}_j^\pm(v)(x) := \begin{cases} \lim\limits_{\substack{y\in T^\mp \\ y\to x}} v(y) & \text{if } n_e^{(j)} < 0, \\ \lim\limits_{\substack{y\in T^\pm \\ y\to x}} v(y) & \text{if } n_e^{(j)} \geq 0, \end{cases} \tag{2.2}$$

$$\mathcal{Q}_j(v)(x) := \frac{1}{2}(\mathcal{Q}_j^-(v)(x) + \mathcal{Q}_j^+(v)(x)) = \{\{v\}\} \tag{2.3}$$

for any $v \in \mathbb{V}^h, x \in e$, and $j = 1, 2, \ldots, d$.

If $e \in \mathcal{E}_h^B$, we set

$$\mathcal{Q}_j^-(v)(x) = \mathcal{Q}_j^+(v)(x) = \mathcal{Q}_j(v)(x) := \lim_{\substack{y\in\Omega \\ y\to x}} v(y) \quad \forall x \in e. \tag{2.4}$$

Remark 2.1.

(i) The functions $\mathcal{Q}_j^-(v)$ and $\mathcal{Q}_j^+(v)$ can be regarded, respectively, as the "left" and "right" limit of v at $x \in e$ in the direction of x_j. Indeed, in the 1D case, we have $\mathcal{Q}_j^\pm(v)(x) = \lim_{y\to x^\pm} v(y)$.

(ii) On an interior edge $e \in \mathcal{E}_h^I$, we may alternatively write

$$\mathcal{Q}_j^\pm(v) = \{\{v\}\} \pm \frac{1}{2}\text{sgn}(n_e^{(j)})[\![v]\!].$$

We shall use this identity frequently in the analysis below.

The discrete second-order differential operators are now simply defined by (2.1), where the trace operator is replaced by $\mathcal{Q}_j^\pm$.

Definition 2.2.

(i) The discrete second differential operators $\partial^{\pm}_{h,i,j}\colon \mathbb{V}^h \to V^h_r$ *are determined by the conditions*

$$(\partial^{\pm}_{h,i,j} v, w)_{\mathcal{T}_h} = \langle \mathcal{Q}^{\pm}_j(\partial_i v) n^{(j)}, [\![w]\!]\rangle_{\mathcal{E}_h} - (\partial_i v, \partial_j w)_{\mathcal{T}_h} \quad \forall w \in V^h_r,$$

where $\partial_i v\colon = \frac{\partial v}{\partial x_i}$ *is the partial derivative of v with respect to* x_i*. We also define the averaged discrete second differential operator* $\partial_{h,i,j} : \mathbb{V}^h \to V^h_r$ *as* $\partial_{h,i,j} := \frac{1}{2}(\partial^{+}_{h,i,j} + \partial^{-}_{h,i,j})$*; that is,*

$$\begin{aligned}(\partial_{h,i,j} v, w)_{\mathcal{T}_h} &= \langle \mathcal{Q}_j(\partial_i v) n^{(j)}, [\![w]\!]\rangle_{\mathcal{E}_h} - (\partial_i v, \partial_j w)_{\mathcal{T}_h} \\ &= \langle \{\{\partial_i v\}\} n^{(j)}, [\![w]\!]\rangle_{\mathcal{E}_h} - (\partial_i v, \partial_j w)_{\mathcal{T}_h}\end{aligned}$$

for all $w \in V^h_r$.

(ii) If boundary data $\boldsymbol{g} = (g^{(1)}, g^{(2)}, \ldots g^{(d)}) \in \boldsymbol{L}^2(\partial\Omega)$ *is given, we set* $\partial^{\pm,g}_{h,i,j} : \mathbb{V}^h \to V^h_r$ *to satisfy*

$$(\partial^{\pm,g}_{h,i,j} v, w)_{\mathcal{T}_h} = (\partial^{\pm}_{h,i,j} v, w)_{\mathcal{T}_h} + \langle (g^{(i)} - \partial_i v) n^{(j)}, w\rangle_{\mathcal{E}^B_h} \quad \forall w \in V^h_r,$$

and define $\partial^{g}_{h,i,j} := \frac{1}{2}(\partial^{+,g}_{h,i,j} + \partial^{-,g}_{h,i,j})$.

(iii) The discrete Hessian operators $\boldsymbol{H}^{\pm}_h, \boldsymbol{H}^{\pm}_{h,g} : \mathbb{V}^h \to \tilde{\boldsymbol{V}}^h_r$ *are defined as*

$$\begin{aligned}(\boldsymbol{H}^{\pm}_h(v))_{i,j} &= \partial^{\pm}_{h,i,j} v, \quad (\boldsymbol{H}_h(v))_{i,j} = \partial_{h,i,j} v, \\ (\boldsymbol{H}^{\pm}_{h,g})_{i,j} &= \partial^{\pm,g}_{h,i,j} v, \quad (\boldsymbol{H}_{h,g})_{i,j} = \partial^{g}_{h,i,j} v.\end{aligned}$$

Remark 2.2. The discrete second-order derivatives and the discrete Hessian defined in Definition 2.2 differ from those given in [7]. In particular, the discrete Hessians in [7] are defined as the composition of discrete first-order derivatives.

Remark 2.3. Since $\mathcal{Q}^{\pm}_j(v) = \{\{v\}\} \pm \frac{1}{2}\mathrm{sgn}(n^{(j)}_e)[\![v]\!]$ on interior edges (cf. Remark 2.1), we have by integration by parts

$$\begin{aligned}(\partial^{\pm,0}_{h,i,j} v, w)_{\mathcal{T}_h} &= \left\langle \{\{\partial_i v\}\} n^{(j)} \pm \frac{1}{2}\mathrm{sgn}(n^{(j)}) n^{(j)} [\![\partial_i v]\!], [\![w]\!]\right\rangle_{\mathcal{E}^I_h} - (\partial_i v, \partial_j w)_{\mathcal{T}_h} \\ &= (\partial_{i,j} v, w)_{\mathcal{T}_h} - \langle [\![\partial_i v]\!] n^{(j)}, \{\{w\}\}\rangle_{\mathcal{E}_h} \pm \frac{1}{2}\langle |n^{(j)}| [\![\partial_i v]\!], [\![w]\!]\rangle_{\mathcal{E}^I_h}.\end{aligned}$$

Consequently, there holds

$$(\boldsymbol{H}^{\pm}_{h,0} v, \boldsymbol{\mu})_{\mathcal{T}_h} = (D^2 v, \boldsymbol{\mu})_{\mathcal{T}_h} - \langle [\![\nabla v]\!] \otimes \boldsymbol{n}, \{\{\mu\}\}\rangle_{\mathcal{E}_h} \pm \frac{1}{2}\langle [\![\nabla v]\!] \otimes |\boldsymbol{n}|, [\![\boldsymbol{\mu}]\!]\rangle_{\mathcal{E}^I_h}$$

for all $\boldsymbol{\mu} \in \tilde{\boldsymbol{V}}^h_r$. Here, $([\![\nabla v]\!] \otimes \boldsymbol{n})_{i,j} = [\![\partial_i v]\!] n^{(j)}$ and $([\![\nabla v]\!] \otimes |\boldsymbol{n}|)_{i,j} = [\![\partial_i v]\!] |n^{(j)}|$ for $i, j = 1, 2, \ldots, d$.

Remark 2.4. The definition of the discrete derivatives is completely local. In particular, $\partial^{\pm}_{h,i,j}v$ on a simplex T only depends on the values of v on all simplices T' with $\partial T \cap T' \neq \emptyset$. To see this, let $w \in V^h_r$ have support only on a single simplex $T \in \mathcal{T}_h$ in Definition 2.2. We then have

$$\int_T (\partial^{\pm}_{h,i,j}v) w dx = \sum_{e \subset \partial T} \int_e \mathcal{Q}^{\pm}_j \left(\frac{\partial v}{\partial x_i} \right) n^{(j)} [\![w]\!] ds - \int_T \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} dx \quad \forall w \in \mathcal{P}_r(T).$$

C. Definition of the Method

The finite element method for the biharmonic problem (1.1) is based on the variational formulation (1.2), where we simply replace the differential operators by a combination of discrete Hessians. To this end, let r and k be two nonnegative integers and define the bilinear form $a_h(\cdot,\cdot) : \mathbb{V}^h \times \mathbb{V}^h \to \mathbb{R}$ as

$$a_h(v,w) := \frac{1}{2}((\boldsymbol{H}^+_{h,0}v, \boldsymbol{H}^+_{h,0}w) + (\boldsymbol{H}^-_{h,0}v, \boldsymbol{H}^-_{h,0}w)),$$

where $\boldsymbol{H}^{\pm}_{h,0}v, \boldsymbol{H}^{\pm}_{h,0}w \in \tilde{\boldsymbol{V}}^h_r$. We then consider the following problem: find a function $u_h \in \mathcal{V}^h_k$ such that

$$a_h(u_h, v) = \langle f, v \rangle \quad \forall v \in \mathcal{V}^h_k. \tag{2.5}$$

Remark 2.5. Set $\boldsymbol{\sigma}^{\pm}_h = \boldsymbol{H}^{\pm}_{h,0}u_h \in \tilde{\boldsymbol{V}}^h_r$. Then, the finite element method (2.5) can be written in the mixed formulation

$$(\boldsymbol{\sigma}^{\pm}_h, \boldsymbol{\mu}) = (D^2 u_h, \boldsymbol{\mu})_{\mathcal{T}_h} - \langle [\![\nabla u_h]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{\mu}\}\} \rangle_{\mathcal{E}_h} \pm \frac{1}{2} \langle [\![\nabla u_h]\!] \otimes |\boldsymbol{n}|, [\![\mu]\!] \rangle_{\mathcal{E}^I_h} \quad \forall \boldsymbol{\mu} \in \tilde{\boldsymbol{V}}^h_r,$$

$$(\boldsymbol{\sigma}^+_h + \boldsymbol{\sigma}^-_h, D^2 v)_{\mathcal{T}_h} - \langle \{\{\boldsymbol{\sigma}^+_h + \boldsymbol{\sigma}^-_h\}\}, [\![\nabla v]\!] \otimes \boldsymbol{n} \rangle_{\mathcal{E}_h} + \frac{1}{2} \langle [\![\boldsymbol{\sigma}^+_h - \boldsymbol{\sigma}^-_h]\!], [\![\nabla v]\!] \otimes |\boldsymbol{n}| \rangle_{\mathcal{E}^I_h} = 2\langle f, v \rangle$$
$$\forall v \in \mathcal{V}^h_k.$$

Equivalent formulations of this mixed system are also possible. For example, if we set $\boldsymbol{\sigma}_h := \frac{1}{2}(\boldsymbol{\sigma}^+_h + \boldsymbol{\sigma}^-_h) = \boldsymbol{H}_{h,0}u_h$ and $\boldsymbol{\tau}_h := \boldsymbol{\sigma}^+_h - \boldsymbol{\sigma}^-_h$, then the method (2.5) reads

$$(\boldsymbol{\sigma}_h, \boldsymbol{\mu}) = (D^2 u_h, \boldsymbol{\mu})_{\mathcal{T}_h} - \langle [\![\nabla u_h]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{\mu}\}\} \rangle_{\mathcal{E}_h} \quad \forall \boldsymbol{\mu} \in \tilde{\boldsymbol{V}}^h_r,$$

$$(\boldsymbol{\tau}_h, \boldsymbol{\mu}) = \langle [\![\nabla u_h]\!] \otimes |\boldsymbol{n}|, [\![\mu]\!] \rangle_{\mathcal{E}^I_h} \quad \forall \boldsymbol{\mu} \in \tilde{\boldsymbol{V}}^h_r,$$

$$(\boldsymbol{\sigma}_h, D^2 v)_{\mathcal{T}_h} - \langle \{\{\boldsymbol{\sigma}_h\}\}, [\![\nabla v]\!] \otimes \boldsymbol{n} \rangle_{\mathcal{E}_h} + \frac{1}{4} \langle [\![\boldsymbol{\tau}_h]\!], [\![\nabla v]\!] \otimes |\boldsymbol{n}| \rangle_{\mathcal{E}^I_h} = \langle f, v \rangle \quad \forall v \in \mathcal{V}^h_k.$$

For comparison, the local-continuous-discontinuous-Galerkin method reads [5]

$$(\boldsymbol{\sigma}_h, \boldsymbol{\mu}) = (D^2 u_h, \boldsymbol{\mu}) - \langle [\![\nabla u_h]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{\mu}\}\} \rangle_{\mathcal{E}_h} \quad \forall \boldsymbol{\mu} \in \tilde{\boldsymbol{V}}^h_r,$$

$$(\boldsymbol{\sigma}_h, D^2 v) - \langle \{\{\boldsymbol{\sigma}_h\}\}, [\![\nabla v]\!] \otimes \boldsymbol{n} \rangle + \langle \alpha_h [\![\nabla u_h]\!], [\![\nabla v]\!] \rangle_{\mathcal{E}_h} = \langle f, v \rangle \quad \forall v \in \mathcal{V}^h_k,$$

where α_h is a piecewise constant penalty parameter. Using the discrete Hessian framework (cf. Remark 2.3), it is easy to see that this method is equivalent to

$$(\boldsymbol{H}_{h,0}u_h, \boldsymbol{H}_{h,0}v) + \langle \alpha_h [\![\nabla u_h]\!], [\![\nabla v]\!] \rangle_{\mathcal{E}_h} = \langle f, v \rangle \quad \forall v \in \mathcal{V}^h_k. \tag{2.6}$$

III. MAIN RESULTS

A. Existence and Uniqueness

The well-posedness of the C^0 finite element method (2.5) is addressed in the next theorem.

Theorem 3.1. *Suppose that $r \geq k-1$ and that each $T \in \mathcal{T}_h$ has at most $(r-k+2)(d-1)$ -dimensional boundary simplices. There exists a unique solution $u_h \in \mathcal{V}_k^h$ to (2.5).*

Proof. Since the method (2.5) represents a square linear system, it suffices to show that if $f=0$, then u_h is identically zero.

Setting $v = u_h$ in (2.5), we obtain $\boldsymbol{H}_{h,0}^{\pm} u_h = 0$. Hence by Remark 2.3, we obtain the identity

$$(\partial_{i,j} u_h, w)_{\mathcal{T}_h} - \langle [\![\partial_i u_h]\!] n^{(j)}, \{\{w\}\}\rangle_{\mathcal{E}_h} \pm \frac{1}{2}\langle |n^{(j)}| [\![\partial_i u_h]\!], [\![w]\!]\rangle_{\mathcal{E}_h^I} = 0 \quad \forall w \in V_r^h \tag{3.1}$$

for $i,j = 1,2,\ldots,d$. Subtracting the above two equations, then yields

$$\langle |n^{(j)}| [\![\partial_i u_h]\!], [\![w]\!]\rangle_{\mathcal{E}_h^I} = 0 \quad \forall w \in V_r^h.$$

Setting $w|_T = \partial_i u_h|_T$ for each $T \in \mathcal{T}_h$, we conclude $[\![\nabla u_h]\!] = 0$ across all interior edges. Therefore, $u_h \in H^2(\Omega) \cap H_0^1(\Omega)$ and (3.1) reads

$$0 = (\partial_{i,j} u_h, w)_{\mathcal{T}_h} - \langle \partial_i u_h n^{(j)}, w\rangle_{\mathcal{E}_h^B}. \tag{3.2}$$

We now construct $w \in V_r^h$ as follows. If $T \in \mathcal{T}_h$ with $\partial T \cap \partial\Omega = \emptyset$, then we set $w|_T \equiv 0$. Otherwise, we denote by $\{e_j\}_{j=0}^m$ with $0 \leq m \leq r-k+1$ the boundary simplices of T, and define $w|_T$ uniquely by the following conditions (cf. Lemma 4.4):

$$\begin{aligned}
\int_T w\kappa\, dx &= \int_T \partial_{i,j} u_h \kappa\, dx \quad \forall \kappa \in \mathcal{P}_{r-m-1}(T),\\
\int_{e_j} w\kappa\, ds &= -\int_{e_j} \partial_i u_h n^{(j)} \kappa\, ds \quad \forall \kappa \in \mathcal{P}_{r-j}(e_j)\ j = 0,1,\ldots,m.
\end{aligned}$$

Note that $\int_T \partial_{i,j} u_h w\, dx = ||\partial_{i,j}^2 u_h||_{L^2(T)}^2$ since $k-2 \leq r-m-1$. Moreover, $\int_{e_j} \partial_i u_h n^{(j)} w\, ds = ||\partial_i u_h||_{L^2(e_j)}^2$ since $k-1 \leq r-m \leq r-j$ for $j \leq m$. Therefore by (3.2), we have

$$\sum_{\substack{T \in \mathcal{T}_h \\ \partial T \cap \partial\Omega \neq \emptyset}} \|D^2 u_h\|_{L^2(T)}^2 + \sum_{e \in \mathcal{E}_h^B} \|\nabla u_h\|_{L^2(e)}^2 = 0.$$

Thus, $u_h \in H_0^2(\Omega)$. By (3.2) once again, we have $(\partial_{i,j} u_h, w)_{\mathcal{T}_h} = 0$ for all $w \in V_r^h$. Choosing $w|_T = \partial_{i,j} u_h|_T$ on each $T \in \mathcal{T}_h$, we conclude that $D^2 u \equiv 0$. Since $u_h \in H_0^2(\Omega)$, this implies $u_h \equiv 0$. ∎

B. Error Estimates

In this section, we state the error estimates of the finite element solution. First by Theorem 3.1, the operator $||\cdot||_h : \mathbb{V} \to \mathbb{R}$ given by

$$||v||_h^2 := a_h(v,v) = \frac{1}{2}(\|\boldsymbol{H}_{h,0}^+ v\|_{L^2(\Omega)}^2 + \|\boldsymbol{H}_{h,0}^- v\|_{L^2(\Omega)}^2)$$

is a norm on $\mathcal{V}_k^h$. Therefore, the proof of [9, Lemma 10.1.9] gives us our starting point in the error analysis.

Corollary 3.1. *The error of the finite element method (2.5) satisfies*

$$||u-u_h||_h \le \inf_{v\in\mathcal{V}_k^h} ||u-v||_h + \sup_{w\in\mathcal{V}_k^h\setminus\{0\}} \frac{a_h(u-u_h,w)}{||w||_h}. \tag{3.3}$$

A few remarks concerning this result are in order. First, we note that $||\cdot||_h$ is not a norm on $H^2(\Omega)+\mathcal{V}_k^h$, and therefore we must establish that the error $||u-u_h||_h$ gives us a meaningful quantity. We also observe that the consistency term in the right-hand side of (3.3) is nonzero since $a_h(u,w) \ne \langle f,w\rangle$ in general. To address these issues and to carry out the convergence analysis below, we shall assume that the solution of the biharmonic problem (1.1) satisfies $u \in H^3(\Omega)$ and $||u||_{H^3(\Omega)} \le C||f||_{H^{-1}(\Omega)}$. In 2D, this elliptic regularity is known to hold provided the domain Ω is convex [10, 11].

Lemma 3.1. *Suppose that the solution to (1.1) satisfies $u \in H^s(\Omega)$ with $s \ge 3$ and that $r \ge k-2$. Define $\ell := \min\{s-2, r+1\}$. Then there holds*

$$\sup_{w\in\mathcal{V}_k^h\setminus\{0\}} \frac{a_h(u-u_h,w)}{||w||_h} \le Ch^\ell ||u||_{H^s(\Omega)} \sup_{w\in\mathcal{V}_k^h\setminus\{0\}} \frac{\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla w]\!]\|_{L^2(e)}^2\right)}{||w||_h}.$$

Lemma 3.2. *Suppose u satisfies $u \in H^s(\Omega)$ with $s \ge 3$ and define $p := \min\{s, k+1\}$. We then have*

$$\inf_{v\in\mathcal{V}_k^h} ||u-v||_h \le Ch^{p-2}||u||_{H^s(\Omega)}.$$

The proofs of Lemmas 3.2 and 3.1 are postponed to the following section. Lemma 3.1 indicates that the error estimate for the finite element method reduces to showing that the induced norm $||\cdot||_h$ intrinsically controls the jump of the gradient, weighted by h_e^{-1}. This issue is addressed in the next crucial lemma. Again, we postpone its proof to the next section.

Lemma 3.3. *Suppose that the integers r and k satisfy $r \ge k-1 \ge 1$. Suppose further that each $T \in \mathcal{T}_h$ has at most $(r-k+2)(d-1)$ -dimensional boundary simplices. Then, there exists a constant $C > 0$, independent of h such that*

$$\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla w]\!]\|_{L^2(e)}^2 \le C||w||_h^2 \quad \forall w \in \mathcal{V}_k^h. \tag{3.4}$$

Theorem 3.2. *Let $u \in H^s(\Omega) \cap H_0^2(\Omega)$ solve the biharmonic problem (1.1) with $s \geq 3$. Let $u_h \in \mathcal{V}_k^h$ be the solution to the finite element method (2.5). Then under the assumptions of Lemma 3.3, there holds*

$$||u - u_h||_h \leq C(h^\ell + h^{p-2})||u||_{H^s(\Omega)}, \tag{3.5}$$

$$||D^2 u - \boldsymbol{H}_{h,0}^{\pm} u_h||_{L^2(\Omega)} \leq C(h^\ell + h^{p-2})||u||_{H^s(\Omega)}, \tag{3.6}$$

$$||u - u_h||_{H^1(\Omega)} \leq C(h^{\ell+1} + h^{p-1})||u||_{H^s(\Omega)}, \tag{3.7}$$

where $\ell = \min\{s-2, r+1\}$ and $p = \min\{s, k+1\}$.

The estimate (3.5) easily follows from Corollary 3.1 and Lemmas 3.2–3.3. The other two estimates are considerably more technical. We give their proofs in Section VD.

Remark 3.1. Theorem 3.2 indicates the requirement $r \geq k-1 \geq 1$ to guarantee the optimal order estimates (3.5)–(3.7). This requirement is due to Lemma 3.3, as Lemmas 3.1–3.2 only require $r \geq k-2 \geq 0$ to achieve optimal order. Of course, one possible remedy is to include penalization terms in the method (2.5), that is, to consider the numerical method

$$a_h(u_h, v) + \langle \alpha_h [\![\nabla u_h]\!], [\![\nabla v]\!] \rangle_{\mathcal{E}_h} = \langle f, v \rangle \quad \forall v \in \mathcal{V}_k^h, \tag{3.8}$$

where $\alpha|_e = \alpha_e h_e^{-1}$ and α_e is a positive constant on $e \in \mathcal{E}_h$. Clearly, with the additional penalization terms Lemma 3.3 holds, and therefore we obtain optimal order estimates with $r \geq k-2 \geq 0$. However, the numerical experiments presented in Section VII indicate that the additional penalization is not needed, and that the method (2.5) satisfies the estimates (3.5)–(3.7) provided $r \geq k-2 \geq 0$.

IV. PRELIMINARY RESULTS

Before proving the results stated in Section III, we first establish a few results concerning the discrete second-order derivatives and integration-by-parts formulas. First, we show that the discrete Hessians acting on smooth functions are simply the L^2-projections.

Lemma 4.1. *If $\varphi \in H_0^2(\Omega)$, then $\boldsymbol{H}_{h,0}^{\pm}\varphi$ is the L^2-projection of the Hessian $D^2\varphi$ onto the finite element space $\tilde{\boldsymbol{V}}_r^h$.*

Proof. Since $[\![\nabla\varphi]\!] = 0$ on all $e \in \mathcal{E}_h$, we have by Remark 2.3

$$(\boldsymbol{H}_{h,0}^{\pm}\varphi - D^2\varphi, \boldsymbol{\mu})_{\mathcal{T}_h} = -\langle [\![\nabla\varphi]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{\mu}\}\} \rangle_{\mathcal{E}_h} \pm \frac{1}{2} \langle [\![\nabla\varphi]\!] \otimes |\boldsymbol{n}|, [\![\mu]\!] \rangle_{\mathcal{E}_h^I} = 0$$

for all $\boldsymbol{\mu} \in \tilde{\boldsymbol{V}}_r^h$. This is the definition of the L^2-projection of $D^2\varphi$. ■

Lemma 4.2. *Suppose that $r \geq k-2$. Then there holds for any $\varphi \in H_0^2(\Omega)$,*

$$(D^2\varphi, D^2 v)_{\mathcal{T}_h} = a_h(\varphi, v) + \langle [\![\nabla v]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{H}_{h,0}\varphi\}\} \rangle_{\mathcal{E}_h} \quad \forall v \in V_k^h. \tag{4.1}$$

Proof. Note that $D^2 V_k^h \subset \tilde{\boldsymbol{V}}_r^h$ provided $r \geq k-2$. Therefore by Lemma 4.1, we have

$$(D^2\varphi, D^2 v)_{\mathcal{T}_h} = \frac{1}{2}(\boldsymbol{H}^+_{h,0}\varphi, D^2 v)_{\mathcal{T}_h} + \frac{1}{2}(\boldsymbol{H}^-_{h,0}\varphi, D^2 v)_{\mathcal{T}_h} \tag{4.2}$$

for all $v \in V_k^h$. Moreover, by Remark 2.3 we have

$$(\boldsymbol{H}^+_{h,0}v, \boldsymbol{H}^+_{h,0}\varphi)_{\mathcal{T}_h} = (D^2 v, \boldsymbol{H}^+_{h,0}\varphi)_{\mathcal{T}_h} - \langle [\![\nabla v]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{H}_{h,0}\varphi\}\}\rangle_{\mathcal{E}_h} + \frac{1}{2}\langle [\![\nabla v]\!] \otimes |\boldsymbol{n}|, [\![\boldsymbol{H}_{h,0}\varphi]\!]\rangle_{\mathcal{E}_h^I}, \tag{4.3}$$

$$(\boldsymbol{H}^-_{h,0}v, \boldsymbol{H}^-_{h,0}\varphi)_{\mathcal{T}_h} = (D^2 v, \boldsymbol{H}^-_{h,0}\varphi)_{\mathcal{T}_h} - \langle [\![\nabla v]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{H}_{h,0}\varphi\}\}\rangle_{\mathcal{E}_h} - \frac{1}{2}\langle [\![\nabla v]\!] \otimes |\boldsymbol{n}|, [\![\boldsymbol{H}_{h,0}\varphi]\!]\rangle_{\mathcal{E}_h^I}, \tag{4.4}$$

where we have used the fact $\boldsymbol{H}^+_{h,0}\varphi = \boldsymbol{H}^-_{h,0}\varphi = \boldsymbol{H}_{h,0}\varphi$. The identity (4.1) now follows from (4.2) to (4.4) and the definition of $a_h(\cdot,\cdot)$. ∎

Lemma 4.3 ([2], Lemma 5). *Suppose the solution to (1.1) satisfies $u \in H^3(\Omega)$. Then, there holds*

$$\langle f, v\rangle = (D^2 u, D^2 v)_{\mathcal{T}_h} - \langle \{\{D^2 u\}\}, [\![\nabla v]\!] \otimes \boldsymbol{n}\rangle_{\mathcal{E}_h} \quad \forall v \in \mathcal{V}_k^h.$$

Lemma 4.4. *Let r and m be two nonnegative integers with $m \leq d$. Let $T \in \mathcal{T}_h$, and let $\{e_j\}_{j=0}^m$ be arbitrary $(d-1)$-dimensional subsimplices of T. Then, any $q \in \mathcal{P}_r(T)$ is uniquely determined by the following values*

$$\int_T q\kappa\, dx \quad \forall \kappa \in \mathcal{P}_{r-m-1}(T), \tag{4.5a}$$

$$\int_{e_j} q\kappa\, ds \quad \forall \kappa \in \mathcal{P}_{r-j}(e_j),\ j = 0, 1, \ldots, m. \tag{4.5b}$$

Here, $\mathcal{P}_s$ denotes the empty set for $s \leq -1$.

Proof. Note that the number of degrees of freedom (DOF) given in (4.5) is

$$\dim \mathcal{P}_{r-m-1}(\mathbb{R}^d) + \sum_{j=0}^m \dim \mathcal{P}_{r-j}(\mathbb{R}^{d-1}) = \binom{r-m-1+d}{d} + \sum_{j=0}^m \binom{r-j+d-1}{d-1} = \binom{r+d}{d} = \dim \mathcal{P}_r(\mathbb{R}^d).$$

Therefore, it suffices to show that $q \in \mathcal{P}_r(T)$ vanishes at the DOFs (4.5) if and only if $q \equiv 0$.

Let $\lambda_j \in \mathcal{P}_1(T)$ be nonnegative functions satisfying $\lambda_j|_{e_j} = 0$ for $j = 0, 1, \ldots, m$. By (4.5b), q vanishes on e_0, and therefore $q = \lambda_0 p_0$ for some $p_0 \in \mathcal{P}_{r-1}(K)$. Since $\lambda_0 > 0$ on e_1, the DOF (4.5b) imply $q=0$ on e_1 as well. Therefore, $q = \lambda_0\lambda_1 p_1$ for some $p_1 \in \mathcal{P}_{r-2}(T)$. Continuing in this fashion, we have $q = (\prod_{j=0}^m \lambda_j) p_m$ for some $p_m \in \mathcal{P}_{r-m-1}(T)$. Finally by (4.5a), we conclude that $q \equiv 0$. ∎

A. The Morley Finite Element Space and its Properties

A key component in the convergence analysis of the finite element method (2.5) is the construction of an operator that connects the Lagrange finite element space with the Morley finite element space. We recall the Morley finite element space, denoted by $M^h \subset V_k^h$, is a nonconforming finite element space for the biharmonic problem. It consists of piecewise quadratic polynomials that are continuous at the following values [12]:

1. $\int_e \frac{\partial v}{\partial n_e} ds$ for all (d − 1)-dimensional simplices e in $\mathcal{T}_h$,
2. $\int_s v da$ for all $(d-2)$-dimensional simplices s in $\mathcal{T}_h$.

For $d = 2$, $\int_s v da = v(s)$, the evaluation of v at a vertex s. In addition, for simplices e and s on the boundary, functions $v \in M^h$ satisfy $\int_e \frac{\partial v}{\partial n_e} ds = 0$ and $\int_s v da = 0$.

Lemma 4.5.

(i) Let $e \in \mathcal{E}_h$ be a (d − 1)-dimensional simplex. Then, there holds

$$\int_e [\![\nabla v]\!] ds = 0 \quad \forall v \in M^h. \tag{4.6}$$

(ii) The following estimates are satisfied for all $v \in M^h$:

$$\sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![\nabla v]\!]\|_{L^2(e)}^2 \le C|v|_{H^2(\mathcal{T}_h)}^2, \tag{4.7}$$

$$||v||_h \le C|v|_{H^2(\mathcal{T}_h)}. \tag{4.8}$$

(iii) There holds for all piecewise constant tensors $\boldsymbol{\mu}$

$$\sum_{T \in \mathcal{T}_h} \int_T (D^2 v - \boldsymbol{H}_{h,0}^{\pm} v) : \boldsymbol{\mu} dx = 0 \quad \forall v \in M^h. \tag{4.9}$$

(iv) For any $v \in M^h$, there exists $v_0 \in \mathcal{V}_k^h$ such that

$$|v - v_0|_{H^m(\mathcal{T}_h)} \le Ch^{2-m}|v|_{H^m(\mathcal{T}_h)} \quad m = 0, 1, 2. \tag{4.10}$$

(v) For any $\varphi \in H^3(\Omega) \cap H_0^2(\Omega)$, there exists $\varphi_h \in M^h$ such that

$$|\varphi - \varphi_h|_{H^m(\mathcal{T}_h)} \le Ch^{3-m}|\varphi|_{H^3(\Omega)} \quad m = 0, 1, 2. \tag{4.11}$$

Proof. Properties (i), (iv), (v), and (4.7) have been reported in [12, Lemmas 3–6]. Therefore, it suffices to show (4.8) and (4.9).

By Remark 2.3, we have for any $\boldsymbol{\mu} \in \tilde{\boldsymbol{V}}_r^h$,

$$(\boldsymbol{H}_{h,0}^{\pm} v, \boldsymbol{\mu})_{\mathcal{T}_h} = (D^2 v, \boldsymbol{\mu})_{\mathcal{T}_h} - \langle [\![\nabla v]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{\mu}\}\} \rangle_{\mathcal{E}_h} \pm \frac{1}{2} \langle [\![\nabla v]\!] \otimes |\boldsymbol{n}|, [\![\mu]\!] \rangle_{\mathcal{E}_h^I}. \tag{4.12}$$

Using the estimate (4.7) and the Cauchy–Schwarz, trace and inverse inequalities, we obtain

$$\langle [\![\nabla v]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{\mu}\}\}\rangle_{\mathcal{E}_h} \le \left(\sum_{e\in\mathcal{E}_h} h_e^{-1} \|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2} \left(\sum_{e\in\mathcal{E}_h} h_e \|\{\{\boldsymbol{\mu}\}\}\|_{L^2(e)}^2\right)^{1/2}$$
$$\le C|v|_{H^2(\mathcal{T}_h)}||\boldsymbol{\mu}||_{L^2(\Omega)}.$$

Similar arguments show $\frac{1}{2}\langle [\![\nabla v]\!]_n \otimes |\boldsymbol{n}|, [\![\boldsymbol{\mu}]\!]\rangle_{\mathcal{E}_h^I} \le C|v|_{H^2(\mathcal{T}_h)}||\boldsymbol{\mu}||_{L^2(\Omega)}$. Applying these estimates to (4.12), we obtain

$$(\boldsymbol{H}_{h,0}^{\pm} v, \boldsymbol{\mu})_{\mathcal{T}_h} \le C|v|_{H^2(\mathcal{T}_h)}||\boldsymbol{\mu}||_{L^2(\Omega)} \quad \forall \boldsymbol{\mu} \in \tilde{\boldsymbol{V}}_r^h.$$

The estimate (4.8) now easily follows by setting $\boldsymbol{\mu} = \boldsymbol{H}_{h,0}^{\pm} v$.

To show property (4.9), we combine (4.12) and (4.6) to obtain

$$\sum_{T\in\mathcal{T}_h} \int_T (D^2 v - \boldsymbol{H}_{h,0}^{\pm} v) : \boldsymbol{\mu} dx = \langle [\![\nabla v]\!] \otimes \boldsymbol{n}, \{\{\boldsymbol{\mu}\}\}\rangle_{\mathcal{E}_h} \mp \frac{1}{2}\langle [\![\nabla v]\!] \otimes |\boldsymbol{n}|, [\![\boldsymbol{\mu}]\!]\rangle_{\mathcal{E}_h^I} = 0.$$

∎

Lemma 4.6. *Suppose that $r \ge k-2 \ge 0$. Then, there exists an operator $\mathcal{I}_h : \mathcal{V}_k^h \to M^h$ such that*

$$|\mathcal{I}_h v|_{H^2(\mathcal{T}_h)} + ||\mathcal{I}_h v||_h \le C||v||_h, \tag{4.13}$$

$$|h_T^{-1}(v - \mathcal{I}_h v)|_{H^1(\mathcal{T}_h)}^2 \le C\left[||v||_h^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right]. \tag{4.14}$$

for all $v \in \mathcal{V}_k^h$. Here, $|h_T^{-1} v|_{H^1(\mathcal{T}_h)}^2 := \sum_{T\in\mathcal{T}_h} h_T^{-2}|v|_{H^1(T)}^2$.

Proof. Given a function $v \in \mathcal{V}_k^h$, define $\mathcal{I}_h v \in M^h$ to be the unique function satisfying

$$(D^2(\mathcal{I}_h v), D^2 w)_{\mathcal{T}_h} = (\boldsymbol{H}_{h,0} v, D^2 w)_{\mathcal{T}_h} \quad \forall w \in M^h. \tag{4.15}$$

By [12, Lemma 8], $\mathcal{I}_h v$ is well-defined.

Setting $w = \mathcal{I}_h v$ in (4.15), we obtain

$$|\mathcal{I}_h v|_{H^2(\mathcal{T}_h)}^2 \le ||\boldsymbol{H}_{h,0} v||_{L^2(\Omega)}|\mathcal{I}_h v|_{H^2(\mathcal{T}_h)} \le ||v||_h |\mathcal{I}_h v|_{H^2(\mathcal{T}_h)}.$$

Dividing by $|\mathcal{I}_h v|_{H^2(\mathcal{T}_h)}$ and using Lemma 4.5.ii gives us the estimate $||\mathcal{I}_h v||_h \le C|\mathcal{I}_h v|_{H^2(\mathcal{T}_h)} \le C||v||_h$. Thus, (4.13) is satisfied.

Next, let $g \in H^{-1}(\Omega)$ be arbitrary, and let $\varphi \in H_0^2(\Omega)$ to be the unique function satisfying the elliptic problem

$$\Delta^2 \varphi = g \quad \text{in } \Omega, \tag{4.16a}$$

$$\varphi = \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{4.16b}$$

By elliptic regularity, we have $\varphi \in H^3(\Omega)$ with $||\varphi||_{H^3(\Omega)} \le C||g||_{H^{-1}(\Omega)}$.

By Lemma 4.5.iv, there exists a function $v_0 \in \mathcal{V}_k^h$ satisfying $|\mathcal{I}_h v - v_0|_{H^m(\mathcal{T}_h)} \le Ch^{2-m}|\mathcal{I}_h v|_{H^2(\mathcal{T}_h)}$ for $m \in \{0,1,2\}$. Multiplying the PDE (4.16a) by $v - v_0$ and integrating by parts, we have

$$\begin{aligned}\langle g, v-v_0\rangle &= -(\nabla\Delta\varphi, \nabla(v-v_0))\\ &= -(\nabla\Delta\varphi, \nabla(v-\mathcal{I}_h v))_{\mathcal{T}_h} - (\nabla\Delta\varphi, \nabla(\mathcal{I}_h v - v_0))_{\mathcal{T}_h}\\ &= -(\nabla\Delta\varphi, \nabla(\mathcal{I}_h v - v_0))_{\mathcal{T}_h}\\ &\quad + \left[(D^2\varphi, D^2(v-\mathcal{I}_h v))_{\mathcal{T}_h} - \langle\{\{D^2\varphi\}\}, [\![\nabla(v-\mathcal{I}_h v)]\!]\otimes \boldsymbol{n}\rangle_{\mathcal{E}_h}\right]\\ &=: J_1 + J_2. \end{aligned} \tag{4.17}$$

By Lemma 4.5 and (4.13), we have

$$J_1 \le |\varphi|_{H^3(\Omega)}|\mathcal{I}_h v - v_0|_{H^1(\mathcal{T}_h)} \le Ch||g||_{H^{-1}(\Omega)}|\mathcal{I}_h v|_{H^2(\mathcal{T}_h)} \le Ch||g||_{H^{-1}(\Omega)}||v||_h. \tag{4.18}$$

Next, since $v - \mathcal{I}_h v \in V_k^h$ and $\boldsymbol{H}_{h,0}\varphi$ is the L^2-projection of $D^2\varphi$ onto $\tilde{\boldsymbol{V}}_r^h$, we have by Lemma 4.2

$$\begin{aligned}(D^2\varphi, D^2(v-\mathcal{I}_h v))_{\mathcal{T}_h} &= (\boldsymbol{H}_{h,0}\varphi, \boldsymbol{H}_{h,0}(v-\mathcal{I}_h v))_{\mathcal{T}_h} + \langle[\![\nabla(v-\mathcal{I}_h v)]\!]\otimes\boldsymbol{n}, \{\{\boldsymbol{H}_{h,0}\varphi\}\}\rangle_{\mathcal{E}_h}\\ &= (D^2\varphi, \boldsymbol{H}_{h,0}(v-\mathcal{I}_h v))_{\mathcal{T}_h} + \langle[\![\nabla(v-\mathcal{I}_h v)]\!]\otimes\boldsymbol{n}, \{\{\boldsymbol{H}_{h,0}\varphi\}\}\rangle_{\mathcal{E}_h}.\end{aligned}$$

Using this identity in the definition of J_2, we obtain

$$\begin{aligned}J_2 &= (D^2\varphi, \boldsymbol{H}_{h,0}(v-\mathcal{I}_h v))_{\mathcal{T}_h} + \langle\{\{\boldsymbol{H}_{h,0}\varphi - D^2\varphi\}\}, \nabla(v-\mathcal{I}_h v)\otimes\boldsymbol{n}\rangle_{\mathcal{E}_h}\\ &=: I_1 + I_2.\end{aligned} \tag{4.19}$$

To derive an upper bound for I_2, we use the Cauchy–Schwarz and trace inequalities and approximation properties of the L^2-projection to obtain

$$\begin{aligned}I_2 &\le \left(\sum_{e\in\mathcal{E}_h} h_e\|\{\{\boldsymbol{H}_{h,0}\varphi - D^2\varphi\}\}\|_{L^2(e)}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v - \mathcal{I}_h v]\!]\|_{L^2(e)}^2\right)^{1/2}\\ &\le Ch||\varphi||_{H^3(\Omega)}\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v - \mathcal{I}_h v]\!]\|_{L^2(e)}^2\right)^{1/2}\\ &\le Ch||g||_{H^{-1}(\Omega)}\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v - \mathcal{I}_h v]\!]\|_{L^2(e)}^2\right)^{1/2}.\end{aligned}$$

Noting that the last term on the right-hand side can be bounded as [cf. (4.7)]

$$\begin{aligned}\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla(v-\mathcal{I}_h v)]\!]\|_{L^2(e)}^2 &\le 2\sum_{e\in\mathcal{E}_h} h_e^{-1}\left[\|[\![\nabla v]\!]\|_{L^2(e)}^2 + \|[\![\nabla(\mathcal{I}_h v))]\!]\|_{L^2(e)}^2\right] \\ &\le C\left[\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2 + |\mathcal{I}_h v|_{H^2(\mathcal{T}_h)}^2\right] \\ &\le C\left[\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2 + ||v||_h^2\right],\end{aligned}$$

we arrive at the following upper bound for I_2:

$$I_2 \le Ch||g||_{H^{-1}(\Omega)}\left[\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2} + ||v||_h\right]. \tag{4.20}$$

To estimate I_1, we let $\varphi_h \in M^h$ satisfy $|\varphi-\varphi_h|_{H^2(\mathcal{T}_h)} \le Ch^{s-2}||\varphi||_{H^s(\Omega)}$ [cf. (4.11)]. We then have

$$\begin{aligned}I_1 &= (D^2(\varphi-\varphi_h), \boldsymbol{H}_{h,0}(v-\mathcal{I}_h v))_{\mathcal{T}_h} + (D^2\varphi_h, \boldsymbol{H}_{h,0}(v-\mathcal{I}_h v))_{\mathcal{T}_h} \\ &= (D^2(\varphi-\varphi_h), \boldsymbol{H}_{h,0}(v-\mathcal{I}_h v))_{\mathcal{T}_h} + (D^2\varphi_h, D^2(\mathcal{I}_h v) - \boldsymbol{H}_{h,0}(\mathcal{I}_h v))_{\mathcal{T}_h} \\ &= (D^2(\varphi-\varphi_h), \boldsymbol{H}_{h,0}(v-\mathcal{I}_h v))_{\mathcal{T}_h} \\ &\le C|\varphi-\varphi_h|_{H^2(\mathcal{T}_h)}||v-\mathcal{I}_h v||_h \le Ch||g||_{H^{-1}(\Omega)}||v||_h.\end{aligned} \tag{4.21}$$

Here, we have used (4.9) and the fact that $D^2\varphi_h$ is piecewise constant.

Applying the estimates (4.20)–(4.21) to (4.19), we obtain

$$J_2 \le Ch^{-1}||g||_{H^{-1}(\Omega)}\left[||v||_h + \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2}\right]. \tag{4.22}$$

Finally, combining the estimates (4.17), (4.18), and (4.22), we have

$$\langle g, v-v_0\rangle \le Ch||g||_{H^{-1}(\Omega)}\left[||v||_h + \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2}\right].$$

Consequently, there holds

$$||v-v_0||_{H^1(\Omega)} \le Ch\left[||v||_h + \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2}\right].$$

The estimate (4.14) now follows from the triangle inequality, the estimate $||\mathcal{I}_h v - v_0||_{H^1(\mathcal{T}_h)} \le Ch|\mathcal{I}_h v|_{H^2(\mathcal{T}_h)} \le Ch||v||_h$, and the quasiuniformity of the mesh. ■

V. PROOFS

A. Proof of Lemma 3.1

We are now in position to prove Lemma 3.1. First by the definition of the finite element scheme, we have $a_h(u-u_h,w)=a_h(u,w)-\langle f,w\rangle$. Therefore, by Lemmas 4.2 and 4.3, we have

$$a_h(u-u_h,w)=\langle\{\{D^2u-\boldsymbol{H}_{h,0}u\}\},[\![\nabla w]\!]\otimes\boldsymbol{n}\rangle_{\mathcal{E}_h}. \tag{5.1}$$

By the Cauchy–Schwarz inequality and approximation properties of the L^2-projection (cf. Lemma 4.1), we have

$$\begin{aligned}a_h(u-u_h,w)&\le\left(\sum_{e\in\mathcal{E}_h}h_e\|\{\{D^2u-\boldsymbol{H}_{h,0}u\}\}\|^2_{L^2(e)}\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[\![\nabla w]\!]\|^2_{L^2(e)}\right)^{1/2}\\&\le Ch^{\ell}||u||_{H^s(\Omega)}\left(\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[\![\nabla w]\!]\|^2_{L^2(e)}\right)^{1/2}\end{aligned}$$

with $\ell=\min\{r+1,s-2\}$. This completes the proof of Lemma 3.1.

B. Proof of Lemma 3.2

To prove Lemma 3.2, we first use the identity given in Remark 2.3 to obtain

$$\begin{aligned}(\boldsymbol{H}^{\pm}_{h,0}(u-v),\boldsymbol{\mu})&=(D^2(u-v),\boldsymbol{\mu})_{\mathcal{T}_h}-\langle[\![\nabla(u-v)]\!]\otimes\boldsymbol{n},\{\{\boldsymbol{\mu}\}\}\rangle_{\mathcal{E}_h}\\&\quad\pm\frac{1}{2}\langle[\![\nabla(u-v)]\!]\otimes|\boldsymbol{n}|,[\![\boldsymbol{\mu}]\!]\rangle_{\mathcal{E}_h}\forall\boldsymbol{\mu}\in\tilde{\boldsymbol{V}}^h_r.\end{aligned}$$

Now let $v\in\mathcal{V}^h_k$ be a function satisfying $|u-v|_{H^m(\mathcal{T}_h)}\le Ch^{p-m}||u||_{H^s(\Omega)}$ for $m=0,1,2$ with $p=\min\{k+1,s\}$ [9, 13]. We then have by a trace inequality and scaling,

$$\begin{aligned}\langle[\![\nabla(u-v)]\!]\otimes\boldsymbol{n},\{\{\boldsymbol{\mu}\}\}\rangle_{\mathcal{E}_h}&\le\left(\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[\![\nabla(u-v)]\!]\|^2_{L^2(e)}\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h}h_e\|\{\{\boldsymbol{\mu}\}\}\|^2_{L^2(e)}\right)^{1/2}\\&\le Ch^{p-2}||u||_{H^s(\Omega)}||\boldsymbol{\mu}||_{L^2(\Omega)}.\end{aligned}$$

Similar arguments show $\frac{1}{2}\langle[\![\nabla(u-v)]\!]\otimes|\boldsymbol{n}|,[\![\boldsymbol{\mu}]\!]\rangle_{\mathcal{E}_h}\le Ch^{p-2}||u||_{H^s(\Omega)}||\boldsymbol{\mu}||_{L^2(\Omega)}$. Consequently, for this choice of v, we have

$$\begin{aligned}||\boldsymbol{H}^{\pm}_{h,0}(u-v)||_{L^2(\Omega)}&=\sup_{\boldsymbol{\mu}\in\tilde{\boldsymbol{V}}^h_r\setminus\{0\}}\frac{(\boldsymbol{H}^{\pm}_{h,0}(u-v),\boldsymbol{\mu})}{||\boldsymbol{\mu}||_{L^2(\Omega)}}\\&\le\sup_{\boldsymbol{\mu}\in\tilde{\boldsymbol{V}}^h_r\setminus\{0\}}\frac{|u-v|_{H^2(\mathcal{T}_h)}||\boldsymbol{\mu}||_{L^2(\Omega)}+Ch^{p-2}||u||_{H^s(\Omega)}||\boldsymbol{\mu}||_{L^2(\Omega)}}{||\boldsymbol{\mu}||_{L^2(\Omega)}}\\&\le Ch^{p-2}||u||_{H^s(\Omega)}.\end{aligned}$$

Lemma 3.2 now follows from this identity and the definition of $||\cdot||_h$.

C. Proof of Lemma 3.3

To show that the norm $||\cdot||_h$ controls the jumps of the gradients across $(d-1)$-dimensional simplices, that is, to show (3.4) holds, we first use the algebraic identity $\frac{1}{2}(a^2+b^2) = \frac{1}{4}(a+b)^2 + \frac{1}{4}(a-b)^2$ to get the following identity

$$||v||_h^2 = \frac{1}{4}||\boldsymbol{H}_{h,0}v||_{L^2(\Omega)}^2 + \frac{1}{4}||\boldsymbol{H}_{h,0}^+ v - \boldsymbol{H}_{h,0}^- v||_{L^2(\Omega)}^2 \quad \forall v \in V_k^h. \tag{5.2}$$

The next lemma shows that the second term in the right-hand side of (5.2) controls the jumps across interior simplices modulo arbitrary small boundary terms.

Lemma 5.1. *Let the integers r and k satisfy $r \geq k-1 \geq 1$. Then for any $v \in \mathcal{V}_k^h$ and any $\tau > 0$, there holds*

$$\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2 \leq C_I(1+\tau^{-1})||v||_h^2 + \tau \sum_{e\in\mathcal{E}_h^B} h_e^{-1}\|\nabla v\|_{L^2(e)}^2,$$

where the constant C_I is independent of h and τ.

Proof. By Remark 2.3, the difference $\boldsymbol{H}_{h,0}^+ v - \boldsymbol{H}_{h,0}^- v$ for a function $v \in V_k^h$ satisfies

$$(\boldsymbol{H}_{h,0}^+ v - \boldsymbol{H}_{h,0}^- v, \boldsymbol{\mu})_{\mathcal{T}_h} = \langle [\![\nabla v]\!] \otimes |\boldsymbol{n}|, [\![\boldsymbol{\mu}]\!]\rangle_{\mathcal{E}_h^I} = \sum_{i,j=1}^{d} \langle [\![\partial_i v]\!] |n^{(j)}|, [\![\mu_{i,j}]\!]\rangle_{\mathcal{E}_h^I} \quad \forall \boldsymbol{\mu} \in \tilde{\boldsymbol{V}}_r^h.$$

Since $r \geq k-1$, we may choose $\boldsymbol{\mu} \in \tilde{\boldsymbol{V}}_r^h$ such that $\mu_{i,j} = h^{-1}\partial_i v$ for $i,j = 1,2,\ldots,d$ on all $T \in \mathcal{T}_h$. We then have by the quasiuniformity of the mesh,

$$(\boldsymbol{H}_{h,0}^+ v - \boldsymbol{H}_{h,0}^- v, \boldsymbol{\mu})_{\mathcal{T}_h} \geq C \sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2.$$

Therefore by duality,

$$||\boldsymbol{H}_{h,0}^+ v - \boldsymbol{H}_{h,0}^- v||_{L^2(\Omega)} = \sup_{\boldsymbol{\mu}\in\tilde{\boldsymbol{V}}_r^h\setminus\{0\}} \frac{(\boldsymbol{H}_{h,0}^+ v - \boldsymbol{H}_{h,0}^- v, \boldsymbol{\mu})}{||\boldsymbol{\mu}||_{L^2(\Omega)}} \geq C\frac{\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2}{|h_{\mathcal{T}}^{-1} v|_{H^1(\mathcal{T}_h)}}.$$

Multiplying this expression by $|h_{\mathcal{T}}^{-1} v|_{H^1(\mathcal{T}_h)}$ and recalling (5.2), we obtain the following trace-type estimate.

$$\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2 \leq C||v||_h |h_{\mathcal{T}}^{-1} v|_{H^1(\mathcal{T}_h)} \quad \forall v \in V_k^h.$$

Next, we replace v with $v - \mathcal{I}_h v \in V_k^h$ in the above expression (this inclusion holds since $k \geq 2$). Using the triangle inequality and Lemma 4.5, we get

$$\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2 \leq 2\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v - \mathcal{I}_h v]\!]\|_{L^2(e)}^2 + 2\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla \mathcal{I}_h v]\!]\|_{L^2(e)}^2$$

$$\leq C\left[|||v - \mathcal{I}_h v||_h |h_{\mathcal{T}}^{-1}(v - \mathcal{I}_h v)|_{H^1(\mathcal{T}_h)} + |\mathcal{I}_h v|_{H^2(\mathcal{T}_h)}^2\right].$$

Applying the estimates (4.13)–(4.14) to this last expression then yields

$$\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2 \le C||v||_h\left[||v||_h + \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2}\right]$$

$$\le C||v||_h\left[||v||_h + \left(\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2} + \left(\sum_{e\in\mathcal{E}_h^B} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2}\right].$$

The desired result now follows from the algebraic identity $ab \le \frac{a^2}{2\varepsilon} + \varepsilon\frac{b^2}{2}$ for any $a, b \in \mathbb{R}$, and $\varepsilon > 0$. ∎

Lemma 5.2. *Let the integers r and k satisfy $r \ge k-1 \ge 1$. Suppose that each $T \in \mathcal{T}_h$ has at most $(r-k+2)(d-1)$ -dimensional boundary simplices. Then there holds for all $v \in \mathcal{V}_k^h$,*

$$\sum_{e\in\mathcal{E}_h^B} h_e^{-1}||\nabla v||_{L^2(e)}^2 \le C_B\left[||v||_h^2 + \sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right], \tag{5.3}$$

where the constant $C_B > 0$ is independent of h.

Proof. We write the L^2-norm of the average discrete Hessian as

$$||\boldsymbol{H}_{h,0}v||_{L^2(\Omega)} = \sup_{\boldsymbol{\mu}\in\tilde{\boldsymbol{V}}_r^h\setminus\{0\}} \frac{(\boldsymbol{H}_{h,0}v, \boldsymbol{\mu})}{||\boldsymbol{\mu}||_{L^2(\Omega)}}. \tag{5.4}$$

Using the definition of the discrete Hessian (cf. Remark 2.3) and applying a trace inequality, we have

$$\begin{aligned}(\boldsymbol{H}_{h,0}v, \boldsymbol{\mu}) &= (D^2v, \boldsymbol{\mu})_{\mathcal{T}_h} - \langle[\![\nabla v]\!]\otimes\boldsymbol{n}, \{\{\boldsymbol{\mu}\}\}\rangle_{\mathcal{E}_h}\\ &\ge (D^2v, \boldsymbol{\mu})_{\mathcal{T}_h} - \langle[\![\nabla v]\!]\otimes\boldsymbol{n}, \{\{\boldsymbol{\mu}\}\}\rangle_{\mathcal{E}_h^B}\\ &\quad - C\left(\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}\right)^{1/2}||\boldsymbol{\mu}||_{L^2(\Omega)} \quad \forall\boldsymbol{\mu}\in\tilde{\boldsymbol{V}}_r^h.\end{aligned}$$

Applying this estimate to (5.4) and recalling (5.2), we obtain

$$||v||_h + C\left(\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2} \ge \sup_{\boldsymbol{\mu}\in\tilde{\boldsymbol{V}}_r^h\setminus\{0\}} \frac{(D^2v, \boldsymbol{\mu})_{\mathcal{T}_h} - \langle\nabla \boldsymbol{v}\otimes\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\mathcal{E}_h^B}}{||\boldsymbol{\mu}||_{L^2(\Omega)}} \quad \forall v\in V_k^h. \tag{5.5}$$

We now construct $\boldsymbol{\mu}\in\tilde{\boldsymbol{V}}_r^h$ as follows. (i) If $T\in\mathcal{T}_h$ with $\partial T\cap\partial\Omega = \emptyset$, then we set $\boldsymbol{\mu}|_T \equiv 0$; (ii) Let T be a simplex in the mesh such that $T\cap\partial\Omega \ne \emptyset$. Denote the $(d-1)$-dimensional boundary simplices of T by $\{e_j\}_{j=0}^m$ with $m \le r-k+1$. We then define $\boldsymbol{\mu}|_T$ uniquely by the following conditions (cf. Lemma 4.4)

$$\int_T \boldsymbol{\mu} : \boldsymbol{\kappa}\, dx = 0 \quad \forall \boldsymbol{\kappa} \in \tilde{\mathcal{P}}_{r-m-1}(T),$$

$$\int_{e_j} \boldsymbol{\mu} : \boldsymbol{\kappa}\, ds = -h_e^{-1} \int_{e_j} \nabla v \otimes \boldsymbol{n}_{e_i} : \boldsymbol{\kappa}\, ds \quad \forall \boldsymbol{\kappa} \in \tilde{\mathcal{P}}_{r-j}(e_j)\, j = 0, 1, \ldots, m.$$

For such a $\boldsymbol{\mu}$, we have by scaling $||\boldsymbol{\mu}||^2_{L^2(T)} \le C \sum_{e\in\mathcal{E}_h^B} h_e^{-1} ||\nabla v||^2_{L^2(e)}$ for all $T \in \mathcal{T}_h$. Therefore, by a trace and inverse inequality,

$$||\boldsymbol{\mu}||_{L^2(\Omega)} \le C|h_{\mathcal{T}}^{-1} v|_{H^1(\mathcal{T}_h)}. \tag{5.6}$$

Moreover, since $r-m-1 \ge k-2$ and $r-j \ge k-1$ for $0 \le j \le m$, we have

$$\begin{aligned}(D^2 v, \boldsymbol{\mu})_{\mathcal{T}_h} - \langle \nabla v \otimes \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\mathcal{E}_h^B} &= -\sum_{e\in\mathcal{E}_h^B} \int_e \nabla v \otimes \boldsymbol{n}_e : \boldsymbol{\mu}\, ds \\ &= \sum_{e\in\mathcal{E}_h^B} h_e^{-1} ||\nabla v \otimes \boldsymbol{n}_e||^2_{L^2(e)} \\ &= \sum_{e\in\mathcal{E}_h^B} h_e^{-1} ||\nabla v||^2_{L^2(e)}. \end{aligned} \tag{5.7}$$

Applying (5.6) and (5.7) to the inequality (5.5) and multiplying the result by $|h_{\mathcal{T}}^{-1} v|_{H^1(\mathcal{T}_h)}$, we find

$$\sum_{e\in\mathcal{E}_h^B} h_e^{-1} ||\nabla v||^2_{L^2(e)} \le C|h_{\mathcal{T}}^{-1} v|_{H^1(\mathcal{T}_h)} \left[||v||_h + \left(\sum_{e\in\mathcal{E}_h^I} h_e^{-1} \|[\![\nabla v]\!]\|^2_{L^2(e)} \right)^{1/2} \right] \quad \forall v \in V_k^h. \tag{5.8}$$

Next, for a function $v \in \mathcal{V}_k^h$, we write

$$\sum_{e\in\mathcal{E}_h^B} h_e^{-1} ||\nabla v||^2_{L^2(e)} \le 2 \sum_{e\in\mathcal{E}_h^B} h_e^{-1} ||\nabla (v - \mathcal{I}_h v)||^2_{L^2(e)} + 2 \sum_{e\in\mathcal{E}_h^B} h_e^{-1} ||\nabla \mathcal{I}_h v||^2_{L^2(e)},$$

where $\mathcal{I}_h$ is operator constructed in Lemma 4.6. Since $\mathcal{I}_h v \in M^h$, we have by Lemmas 4.5 and 4.6,

$$\sum_{e\in\mathcal{E}_h^B} h_e^{-1} ||\nabla \mathcal{I}_h v||^2_{L^2(e)} \le C|\mathcal{I}_h v|^2_{H^2(\mathcal{T}_h)} \le C||v||_h^2.$$

By Lemmas 4.5–4.6 and the estimate (5.8) with v replaced by $v - \mathcal{I}_h v$, we also find

$$\begin{aligned}&\sum_{e\in\mathcal{E}_h^B} h_e^{-1} ||\nabla (v - \mathcal{I}_h v)||^2_{L^2(e)} \\ &\quad \le C|h_{\mathcal{T}}^{-1} (v - \mathcal{I}_h v)|_{H^1(\mathcal{T}_h)} \left[||v - \mathcal{I}_h v||_h + \left(\sum_{e\in\mathcal{E}_h^I} h_e^{-1} \|[\![\nabla (v - \mathcal{I}_h v)]\!]\|^2_{L^2(e)} \right)^{1/2} \right]\end{aligned}$$

$$\leq C|h_{\mathcal{T}}^{-1}(v-\mathcal{I}_h v)|_{H^1(\mathcal{T}_h)}\left[\|v\|_h+|\mathcal{I}_h v|_{H^2(\mathcal{T}_h)}+\left(\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2}\right]$$

$$\leq C\left[\|v\|_h+\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right)^{1/2}\right]\left[\|v\|_h+\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right]$$

$$\leq (C+\tau^{-1})\left[\|v\|_h^2+\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right]+\tau\sum_{e\in\mathcal{E}_h^B} h_e^{-1}\|\nabla v\|_{L^2(e)}^2.$$

Here, we have used the inequality $ab \leq \frac{a}{2\varepsilon}+\varepsilon\frac{b}{2}$ to derive the last inequality.

Combining the above estimates, we have

$$\sum_{e\in\mathcal{E}_h^B} h_e^{-1}\|\nabla v\|_{L^2(e)}^2 \leq C\|v\|_h^2+(C+\tau^{-1})\left[\|v\|_h^2+\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2\right]+\tau\sum_{e\in\mathcal{E}_h^B} h_e^{-1}\|\nabla v\|_{L^2(e)}^2$$

for all $v\in\mathcal{V}_k^h$ and $\tau>0$. Taking τ sufficiently small, we obtain (5.3). ■

Proof of Lemma 3.3. Combining Lemmas 5.1 and 5.2, we have for any $\tau>0$

$$\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2 \leq C_I(1+\tau^{-1})\|v\|_h^2+\tau\sum_{e\in\mathcal{E}_h^B} h_e^{-1}\|\nabla v\|_{L^2(e)}^2$$

$$\leq [C_I(1+\tau^{-1})+\tau C_B]\|v\|_h^2+\tau C_B\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2.$$

Taking τ sufficiently small, we obtain $\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![\nabla v]\!]\|_{L^2(e)}^2 \leq C\|v\|_h^2$. Lemma 3.3 now follows from this estimate and Lemma 5.2.

D. Proof of Theorem 3.2

In this section, we prove the error estimates given in Theorem 3.2. Since (3.5) easily follows from Corollary 3.1 and Lemmas 3.2–3.3, we focus our attention on the other two estimates (3.6) and (3.7).

First, since $\boldsymbol{H}_{h,0}^{\pm}u$ is the L^2-projection of the Hessian D^2u onto the finite element space $\tilde{\boldsymbol{V}}_r^h$, we have by (3.5)

$$\|D^2u-\boldsymbol{H}_{h,0}^{\pm}u_h\|_{L^2(\Omega)} \leq \|u-u_h\|_h+\|D^2u-\boldsymbol{H}_{h,0}^{\pm}u\|_{L^2(\Omega)} \leq C(h^{\ell}+h^{p-2})\|u\|_{H^s(\Omega)}.$$

This establishes the estimate (3.6).

To show the H^1-estimate, we use a duality argument. To this end, let $\varphi\in H^3(\Omega)\cap H_0^2(\Omega)$ satisfy the auxiliary problem (4.16) for some $g\in H^{-1}(\Omega)$. By (5.1) we have

$$a_h(\varphi,w)-\langle g,w\rangle=\langle\{\{D^2\varphi-\boldsymbol{H}_{h,0}\varphi\}\},[\![\nabla w]\!]\otimes\boldsymbol{n}\rangle_{\mathcal{E}_h}\quad\forall w\in\mathcal{V}_k^h.$$

Therefore, we find

$$\begin{aligned}\langle g, u-u_h\rangle &= (D^2\varphi, D^2u) - a_h(\varphi, u_h) + \langle\{\{D^2\varphi - \boldsymbol{H}_{h,0}\varphi\}\}, [\![\nabla u_h]\!]\otimes \boldsymbol{n}\rangle_{\mathcal{E}_h}\\ &= [(D^2\varphi, D^2u) - a_h(\varphi, u)] + a_h(\varphi, u-u_h) + \langle\{\{D^2\varphi - \boldsymbol{H}_{h,0}\varphi\}\}, [\![\nabla u_h]\!]\otimes \boldsymbol{n}\rangle_{\mathcal{E}_h}\\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{5.9}$$

To estimate I_1, we use the fact that $\boldsymbol{H}^{\pm}_{h,0}u$ and $\boldsymbol{H}^{\pm}_{h,0}\varphi$ are the L^2-projections of D^2u and $D^2\varphi$, respectively, to obtain

$$I_1 = (D^2\varphi, D^2u) - (\boldsymbol{H}_{h,0}\varphi, D^2u) = (D^2\varphi - \boldsymbol{H}_{h,0}\varphi, D^2u - \boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in \tilde{\boldsymbol{V}}^h_r.$$

Judiciously choosing $\boldsymbol{\mu}$ we obtain

$$\begin{aligned} I_1 &\le ||D^2\varphi - \boldsymbol{H}_{h,0}\varphi||_{L^2(\Omega)}||D^2u - \boldsymbol{\mu}||_{L^2(\Omega)}\\ &\le Ch^{\ell+1}||\varphi||_{H^3(\Omega)}||u||_{H^s(\Omega)} \le Ch^{\ell+1}||g||_{H^{-1}(\Omega)}||u||_{H^s(\Omega)}. \end{aligned} \tag{5.10}$$

To estimate I_2, we let $\varphi_h \in \mathcal{V}^h_k$ be a function satisfying $\sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![\nabla\varphi_h]\!]||^2_{L^2(e)} + ||\varphi - \varphi_h||_h^2 \le Ch^2||\varphi||^2_{H^3(\Omega)}$; see the proof of Lemma 3.2. We then have by (5.1) and scaling,

$$\begin{aligned} I_2 &= a_h(u-u_h, \varphi-\varphi_h) + a_h(u-u_h, \varphi_h)\\ &= a_h(u-u_h, \varphi-\varphi_h) + \langle\{\{D^2u - \boldsymbol{H}_{h,0}u\}\}, [\![\nabla\varphi_h]\!]\otimes\boldsymbol{n}\rangle_{\mathcal{E}_h}\\ &\le ||u-u_h||_h||\varphi-\varphi_h||_h + \left(\sum_{e\in\mathcal{E}_h}\|\{\{D^2u - \boldsymbol{H}_{h,0}u\}\}\|^2_{L^2(e)}\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h}\|[\![\nabla\varphi_h]\!]\|^2_{L^2(e)}\right)^{1/2}\\ &\le C(h^{\ell+1} + h^{p-1})||\varphi||_{H^3(\Omega)}||u||_{H^s(\Omega)} \le C(h^{\ell+1} + h^{p-1})||g||_{H^{-1}}||u||_{H^s(\Omega)}. \end{aligned} \tag{5.11}$$

By the Cauchy–Schwarz inequality and scaling, we have

$$\begin{aligned} I_3 &\le \left(\sum_{e\in\mathcal{E}_h} h_e\|\{\{D^2\varphi - \boldsymbol{H}_{h,0}\varphi\}\}\|^2_{L^2(e)}\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla u_h]\!]\|^2_{L^2(e)}\right)^{1/2}\\ &\le Ch\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla u_h]\!]\|^2_{L^2(e)}\right)^{1/2}||g||_{H^{-1}(\Omega)}. \end{aligned} \tag{5.12}$$

Let $v \in \mathcal{V}^h_k$ satisfy $||u - v||_h \le Ch^{p-2}||u||_{H^s(\Omega)}$ and $|u - v|_{H^m(\mathcal{T}_h)} \le Ch^{p-m}||u||_{H^s(\Omega)}$ for $m = 0, 1, 2$, and $p = \min\{k+1, s\}$; see Lemma 3.2 and its proof. By the triangle inequality, Lemma 3.3, scaling and (3.5), we have

$$\begin{aligned} \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla u_h]\!]\|^2_{L^2(e)}\right)^{1/2} &\le \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla(u_h - v)]\!]\|^2_{L^2(e)}\right)^{1/2} + \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\nabla(v-u)]\!]\|^2_{L^2(e)}\right)^{1/2}\\ &\le C(||u_h - v||_h + h^{p-2}||u||_{H^s(\Omega)})\\ &\le C(||u - u_h||_h + ||u - v||_h + h^{p-2}||u||_{H^s(\Omega)})\\ &\le C(h^{\ell} + h^{p-2})||u||_{H^s(\Omega)}. \end{aligned}$$

Applying this estimate to (5.12), we obtain

$$I_3 \le C\left(h^{\ell+1} + h^{p-1}\right) ||u||_{H^s(\Omega)} ||g||_{H^{-1}(\Omega)}. \tag{5.13}$$

Therefore by (5.10)–(5.11), (5.13), and (5.9), we have

$$\langle g, u - u_h \rangle \le C(h^{\ell+1} + h^{p-1}) ||u||_{H^s(\Omega)} ||g||_{H^{-1}(\Omega)}.$$

Since $g \in H^{-1}(\Omega)$ was arbitrary, we have $||u - u_h||_{H^1(\Omega)} \le C(h^{\ell+1} + h^{p-1})||u||_{H^s(\Omega)}$. The proof is complete.

VI. EXTENSION TO THE KIRCHHOFF PLATES

In this section, we describe how the framework presented in Section II may be applied to the clamped Kirchhoff plate model:

$$\nabla \cdot (\nabla \cdot \mathcal{M}(u)) + f = 0 \quad \text{in } \Omega, \tag{6.1a}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{6.1b}$$

Here, f represents the given vertical load of the plate,

$$\mathcal{M}(u) := D[(\nu - 1)D^2 u - \nu \Delta u I_{2\times 2}],$$

are the bending moments, $\nu \in (0, 0.5)$ denote the Poisson ratio, $D := Eh^3/(12(1 - \nu^2))$ is the (isotropic) plate rigidity, E is elastic modulus of the material, and h is the plate thickness. For simplicity, we assume $D \equiv 1$ in the discussion below. Following the framework given in [5], we reformulate (6.1) as the second-order system

$$\frac{1}{1-\nu}\boldsymbol{\sigma} - \frac{\nu}{1-\nu^2}\text{tr}(\boldsymbol{\sigma}) I_{2\times 2} = -D^2 u \quad \text{in } \Omega, \tag{6.2a}$$

$$\nabla \cdot (\nabla \cdot \boldsymbol{\sigma}) = -f \quad \text{in } \Omega, \tag{6.2b}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{6.2c}$$

where $\text{tr}(\boldsymbol{\sigma}) := \sigma_{11} + \sigma_{22}$. Using simple integration-by-parts formulas, the variational formulation for this system is given by

$$\frac{1}{1-\nu}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\mu} dx - \frac{\nu}{1-\nu^2}\int_\Omega \text{tr}(\boldsymbol{\sigma})\text{tr}(\boldsymbol{\mu}) dx = -\int_\Omega D^2 u : \boldsymbol{\mu} dx \quad \forall \boldsymbol{\mu} \in \tilde{\boldsymbol{L}}^2(\Omega),$$

$$\int_\Omega \boldsymbol{\sigma} : D^2 v dx = -\int_\Omega f v dx \quad \forall v \in H_0^2(\Omega).$$

The proposed numerical method is based on this variational formulation, where the Hessian matrices are replaced by their discrete versions. To this end, we first introduce two auxiliary variables, approximations to the moment tensor (6.2a):

$$\boldsymbol{\sigma}_h^{\pm} := (\nu - 1)\boldsymbol{H}_{h,0}^{\pm} u_h - \nu \text{tr}(\boldsymbol{H}_{h,0}^{\pm} u_h) I_{2\times 2} \in \tilde{\boldsymbol{V}}_r^h. \tag{6.3}$$

The C^0 finite element method for (6.1) is then defined as seeking $u_h \in \mathcal{V}_h$ such that

$$\frac{1}{2}[(\boldsymbol{\sigma}_h^+, \boldsymbol{H}_{h,0}^+ v) + (\boldsymbol{\sigma}_h^-, \boldsymbol{H}_{h,0}^- v)] = -\langle f, v\rangle \quad \forall v \in \mathcal{V}_k^h.$$

Equivalently, the primal formulation is given by

$$a_h(u_h, v) := (1-\nu)(\boldsymbol{H}_{h,0}^+ u_h, \boldsymbol{H}_{h,0}^+ v) + \nu(\operatorname{tr}(\boldsymbol{H}_{h,0}^+ u_h), \operatorname{tr}(\boldsymbol{H}_{h,0}^+ v)) + (1-\nu)(\boldsymbol{H}_{h,0}^- u_h, \boldsymbol{H}_{h,0}^- v) + \nu(\operatorname{tr}(\boldsymbol{H}_{h,0}^- u_h), \operatorname{tr}(\boldsymbol{H}_{h,0}^- v)) = 2\langle f, v\rangle \quad \forall v \in \mathcal{V}_k^h. \tag{6.4}$$

Clearly, we have $a_h(v, v) \geq 2(1-\nu)||v||_h^2$ for all $v \in \mathcal{V}_k^h$. As such, all of the results stated in Section III apply. In particular, the solution to (6.4) satisfies the estimates (3.5)–(3.7).

VII. NUMERICAL EXPERIMENTS

A. Test 1

In this test, we perform some simple numerical experiments which show that the finite element method (2.5) converges optimally in the energy norm. The numerical runs are performed on the unit square $\Omega = (0,1)^2$, and the data are chosen such that the exact solution is given by $u = \sin^2(2\pi x_1)\sin^2(2\pi x_2) \in C^\infty(\overline{\Omega})$. The resulting errors are listed in Table I below with polynomial degree $k \in \{2, 3\}$ and $k-2 \leq r \leq k$. The table clearly shows that the discrete Hessians converge to $D^2 u$ with order $\mathcal{O}(h^{k-1})$ in all cases, where as the H^1 and L^2 errors converge with order $\mathcal{O}(h^3)$ and $\mathcal{O}(h^4)$, respectively, when $k=3$. The numerical experiments also show in the quadratic case that the error converges optimally in the H^1-norm, but suboptimal in the L^2-norm (by one power). These rates are similar to other (primal) methods for the biharmonic problem (e.g., [2, 14, 15]).

B. Test 2

For the second set of experiments, we show by way of numerical example that the $H^3(\Omega)$ regularity assumption can likely be relaxed in the convergence analysis. To this end, we compute the finite element method (2.5) on the L-shaped domain $\Omega = (-1,1)^2 \backslash (0,1) \times (-1,0)$ and choose the data such that exact solution is given by [3, 10]

$$u = r^{1+\alpha} g(\theta),$$

where

$$g(\theta) = g_1(1.5\pi)g_2(\theta) - g_1(\theta)g_2(1.5\pi),$$
$$g_1(\theta) = \frac{1}{\alpha-1}\sin((\alpha-1)\theta) - \frac{1}{\alpha+1}\sin((\alpha+1)\theta),$$
$$g_2(\theta) = \cos((\alpha-1)\theta) - \cos((\alpha+1)\theta),$$

and $\alpha = 0.544483736782464$ is (an approximation of) the noncharacteristic root of $\sin^2(1.5\pi\alpha) = \alpha^2 \sin^2(1.5\pi)$. The resulting errors for various values of h are reported in Table II

TABLE I. Resulting errors of the finite element method (2.5) with exact solution $u = \sin^2(2\pi x_1)\sin^2(2\pi x_2)$.

	h	$\|\|u - u_h\|\|_{L^2}$	Rate	$\|\|\nabla(u - u_h)\|\|_{L^2}$	Rate	$\|\|D^2 u - \boldsymbol{H}_{h,0}^{+} u_h\|\|_{L^2}$	Rate	$\|\|D^2 u - \boldsymbol{H}_{h,0}^{-} u_h\|\|_{L^2}$	Rate
$k = 2$	1/4	1.58E-01		1.90E+00		3.81E+01		3.91E+01	
$r = 2$	1/8	6.24E-02	1.34	8.18E-01	1.22	2.72E+01	0.49	2.65E+01	0.56
	1/16	1.88E-02	1.73	2.45E-01	1.74	1.54E+01	0.82	1.44E+01	0.88
	1/32	5.17E-03	1.86	6.66E-02	1.88	8.11E+00	0.92	7.52E+00	0.94
	1/64	1.35E-03	1.94	1.72E-02	1.95	4.13E+00	0.97	3.83E+00	0.97
$k = 2$	1/4	1.01E-01		1.45E+00		2.99E+01		3.49E+01	
$r = 1$	1/8	4.56E-02	1.15	6.35E-01	1.19	2.13E+01	0.49	2.36E+01	0.56
	1/16	1.44E-02	1.66	1.92E-01	1.73	1.24E+01	0.79	1.28E+01	0.89
	1/32	3.93E-03	1.87	5.19E-02	1.89	6.53E+00	0.92	6.58E+00	0.96
	1/64	1.02E-03	1.95	1.33E-02	1.96	3.32E+00	0.97	3.32E+00	0.98
$k = 2$	1/4	2.48E-01		1.88E+00		4.34E+01		4.12E+01	
$r = 0$	1/8	2.69E-02	3.20	5.64E-01	1.73	2.73E+01	0.67	2.76E+01	0.58
	1/16	6.36E-03	2.08	1.71E-01	1.72	1.48E+01	0.88	1.54E+01	0.84
	1/32	2.57E-03	1.31	4.49E-02	1.93	7.10E+00	1.06	7.30E+00	1.07
	1/64	7.58E-04	1.76	1.11E-02	2.01	3.28E+00	1.12	3.31E+00	1.14
$k = 3$	1/4	3.18E-02		4.40E-01		1.76E+01		1.79E+01	
$r = 3$	1/8	2.48E-03	3.68	5.74E-02	2.94	4.98E+00	1.82	5.22E+00	1.78
	1/16	1.57E-04	3.99	6.60E-03	3.12	1.25E+00	1.99	1.34E+00	1.96
	1/32	1.00E-05	3.96	8.21E-04	3.01	3.19E-01	1.97	3.44E-01	1.96
	1/64	6.33E-07	3.99	1.02E-04	3.01	8.04E-02	1.99	8.67E-02	1.99
$k = 3$	1/4	2.91E-02		4.08E-01		1.76E+01		1.79E+01	
$r = 2$	1/8	2.01E-03	3.86	4.84E-02	3.07	4.74E+00	1.89	5.22E+00	1.78
	1/16	1.19E-04	4.08	5.36E-03	3.17	1.15E+00	2.05	1.33E+00	1.97
	1/32	7.44E-06	4.00	6.53E-04	3.04	2.88E-01	1.99	3.40E-01	1.97
	1/64	4.64E-07	4.00	8.04E-05	3.02	7.24E-02	1.99	8.58E-02	1.99
$k = 3$	1/4	2.98E-02		7.91E-01		2.16E+01		2.29E+01	
$r = 1$	1/8	1.70E-03	4.13	7.01E-02	3.50	6.74E+00	1.68	6.84E+00	1.74
	1/16	8.44E-05	4.34	6.64E-03	3.40	1.74E+00	1.95	1.72E+00	1.99
	1/32	4.30E-06	4.29	6.48E-04	3.36	4.42E-01	1.98	4.32E-01	2.00
	1/64	2.28E-07	4.24	6.55E-05	3.31	1.11E-01	1.99	1.09E-01	1.99

TABLE II. Resulting errors of the finite element method (2.5) with exact solution $u = r^{1+\alpha} g(\theta)$ on an L-shaped domain.

	h	$\|\|u - u_h\|\|_{L^2}$	Rate	$\|\|\nabla(u - u_h)\|\|_{L^2}$	Rate	$\|\|D^2u - \boldsymbol{H}^+_{h,0}u_h\|\|_{L^2}$	Rate	$\|\|D^2u - \boldsymbol{H}^-_{h,0}u_h\|\|_{L^2}$	Rate
$k = 2$	1/4	1.47E-02		7.19E-02		2.05E+00		1.88E+00	
$r = 2$	1/8	7.85E-03	0.90	3.54E-02	1.02	1.39E+00	0.56	1.29E+00	0.55
	1/16	4.04E-03	0.96	1.79E-02	0.98	9.63E-01	0.53	8.89E-01	0.53
	1/32	1.95E-03	1.05	8.64E-03	1.05	6.61E-01	0.54	6.12E-01	0.54
	1/64	9.31E-04	1.07	4.11E-03	1.07	4.54E-01	0.54	4.19E-01	0.54
$k = 2$	1/4	3.65E-03		4.69E-02		1.65E+00		1.63E+00	
$r = 1$	1/8	8.56E-04	2.09	1.64E-02	1.51	1.13E+00	0.55	1.14E+00	0.52
	1/16	5.72E-04	0.58	6.15E-03	1.42	7.81E-01	0.53	7.90E-01	0.53
	1/32	3.18E-04	0.85	2.38E-03	1.37	5.39E-01	0.54	5.44E-01	0.54
	1/64	1.65E-04	0.95	9.82E-04	1.28	3.69E-01	0.54	3.74E-01	0.54
$k = 2$	1/4	3.52E-02		1.66E-01		2.07E+00		2.29E+00	
$r = 0$	1/8	1.48E-02	1.25	7.24E-02	1.20	1.47E+00	0.49	1.63E+00	0.49
	1/16	6.25E-03	1.24	3.10E-02	1.22	1.04E+00	0.50	1.13E+00	0.53
	1/32	2.76E-03	1.18	1.34E-02	1.21	7.20E-01	0.53	7.80E-01	0.54
	1/64	1.26E-03	1.13	5.91E-03	1.18	4.97E-01	0.53	5.36E-01	0.54

with $k = 2$ and $r \in \{0, 1, 2\}$. Similar to the previous test, the rates of convergence differ little with respect to r. In all three cases, we observe the rates of convergence

$$||u - u_h||_{L^2(\Omega)}, \quad ||\nabla(u - u_h)||_{L^2(\Omega)} \approx \mathcal{O}(h^{2\alpha}), \quad ||u - \boldsymbol{H}_h^{\pm} u_h||_h \approx \mathcal{O}(h^{\alpha}).$$

Since the exact solution satisfies $u \in H^{2+\alpha}(\Omega)$, these are the expected rates.

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