ORIGINAL PAPER

ON THE SOLVABLE OF NONLINEAR DIFFERENCE EQUATION OF SIXTH-ORDER

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Abstract. *In this paper, we deal with the next difference equation*

$$x_n = \frac{x_{n-4}x_{n-5}x_{n-6}}{x_{n-1}x_{n-2}\left(a+bx_{n-3}x_{n-4}x_{n-5}x_{n-6}\right)},\ n\in\mathbb{N}_0,$$

where $a, b\in\mathbb{R}$ *and the initial conditions* $x_{-i},\ i\in\{1,2,\ldots,6\}$, *are real numbers. Also, we give the form of well-defined solution of aforementioned difference equation. Finally, we study the asymptotic behavior of well-defined solution of this equation by using obtained formulas.*

Keywords: *Difference equation, periodic solution, well-defined solution.*

1. INTRODUCTION

Our aim in this study is to show that the following difference equation

$$x_n = \frac{x_{n-4}x_{n-5}x_{n-6}}{x_{n-1}x_{n-2}\left(a+bx_{n-3}x_{n-4}x_{n-5}x_{n-6}\right)},\ n\in\mathbb{N}_0, \tag{1.1}$$

where $a, b\in\mathbb{R}$ and the initial conditions $x_{-i},\ i\in\{1,2,\ldots,6\}$, are real numbers, is solvable in closed-form by using suitable transformation. Also, it is to study the asymptotic behavior of well-defined solution of Eq. (1.1).

In recent years, due to important the theory and applications of difference equations and system of difference equations to physics, economy, ecology, biology and so forth, one can see the citation in [1-11], to find the difference equation or their system which can be solved in closed form have been studied extensively in the following articles [12-30]. Besides, it is very important to characterize the behavior of the solutions of these equations and systems. Although many methods are proposed by researchers, the most basic method to do this is to find a closed formula of the solution of the equation or system and analyze it. For example, Tollu et al. [23] studied the solutions of two specials types of Riccati difference equations

$$x_{n+1} = \frac{1}{1+x_n} \text{ and } y_{n+1} = \frac{1}{-1+y_n},\ n\in\mathbb{N}_0,$$

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such that the obtained solutions are related to Fibonacci numbers. Cinar, in [13, 31], got the form of solutions of the next difference equations

$$x_{n+1}=\frac{ax_{n-1}}{1+bx_nx_{n-1}} \text{ and } x_{n+1}=\frac{x_{n-1}}{-1+ax_nx_{n-1}}.$$

Elsayed, in [14, 16], investigated the behavior and expression of the solutions of the following difference equations

$$x_{n+1}=\frac{x_{n-2}x_{n-4}}{x_{n-1}\left(\pm1\pm x_{n-2}x_{n-4}\right)} \text{ and } x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n\left(\pm1\pm x_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)}.$$

Touafek et al. [25], studied the solution form of the next difference equations systems

$$x_{n+1}=\frac{x_{n-2}x_{n-4}}{x_{n-1}\left(\pm1\pm x_{n-2}x_{n-4}\right)} \text{ and } x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n\left(\pm1\pm x_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)}.$$

The paper is organized as follows. In section 2, we will give the solution form of Eq. (1.1) by using suitable transformation. Later, we will study the asymptotic behavior of well-defined solution of Eq. (1.1) according to the parameters *a* and *b*. In section 3 we will present numerical examples for the solution of Eq. (1.1) regarded to the different values *a* and *b*. The obtained solutions are plotted in (n, x_n)-plane by using Mathematica 9.

2. MAIN RESULTS

The aim of this section is to show that Eq. (1.1) can be solved in closed form. Another aim of this section is to investigate the behavior of the solution of Eq. (1.1). Throughout this paper we suppose that the general solution of Eq. (1.1) is well-defined.

Theorem 2.1. Suppose that $(x_n)_{n\geq-6}$ is a well-defined solution of Eq. (1.1). Then, for $m\geq-1$ and $j\in\{2,3,4,5\}$, we have

$$x_{12m+3j}=x_{3j-12}\prod_{s=0}^{m}\frac{bX_{-1}+(1-a-bX_{-1})a^{4s+j}}{bX_{-3}+(1-a-bX_{-3})a^{4s+j+1}}\frac{bX_{-2}+(1-a-bX_{-2})a^{4s+j-1}}{bX_{-1}+(1-a-bX_{-1})a^{4s+j-1}}\frac{bX_{-3}+(1-a-bX_{-3})a^{4s+j-2}}{bX_{-2}+(1-a-bX_{-2})a^{4s+j-2}},$$

$$x_{12m+3j+1}=x_{3j-11}\prod_{s=0}^{m}\frac{bX_{-3}+(1-a-bX_{-3})a^{4s+j+1}}{bX_{-2}+(1-a-bX_{-2})a^{4s+j+1}}\frac{bX_{-1}+(1-a-bX_{-1})a^{4s+j-1}}{bX_{-3}+(1-a-bX_{-3})a^{4s+j}}\frac{bX_{-2}+(1-a-bX_{-2})a^{4s+j-2}}{bX_{-1}+(1-a-bX_{-1})a^{4s+j-2}},$$

$$x_{12m+3j+2}=x_{3j-10}\prod_{s=0}^{m}\frac{bX_{-2}+(1-a-bX_{-2})a^{4s+j+1}}{bX_{-1}+(1-a-bX_{-1})a^{4s+j+1}}\frac{bX_{-3}+(1-a-bX_{-3})a^{4s+j}}{bX_{-2}+(1-a-bX_{-2})a^{4s+j}}\frac{bX_{-1}+(1-a-bX_{-1})a^{4s+j-2}}{bX_{-3}+(1-a-bX_{-3})a^{4s+j-1}},$$

If $a \neq 1$, and

$$x_{12m+3j} = x_{3j-12} \prod_{s=0}^{m} \frac{1+(4s+j)bX_{-1}}{1+(4s+j+1)bX_{-3}} \frac{1+(4s+j-1)bX_{-2}}{1+(4s+j-1)bX_{-1}} \frac{1+(4s+j-2)bX_{-3}}{1+(4s+j-2)bX_{-2}},$$

$$x_{12m+3j+1} = x_{3j-11} \prod_{s=0}^{m} \frac{1+(4s+j+1)bX_{-3}}{1+(4s+j+1)bX_{-2}} \frac{1+(4s+j-1)bX_{-1}}{1+(4s+j)bX_{-3}} \frac{1+(4s+j-2)bX_{-2}}{1+(4s+j-2)bX_{-1}},$$

$$x_{12m+3j+2} = x_{3j-10} \prod_{s=0}^{m} \frac{1+(4s+j+1)bX_{-2}}{1+(4s+j+1)bX_{-1}} \frac{1+(4s+j)bX_{-3}}{1+(4s+j)bX_{-2}} \frac{1+(4s+j-2)bX_{-1}}{1+(4s+j-1)bX_{-3}},$$

If $a = 1$, where $X_{-1} = x_{-1}x_{-2}x_{-3}x_{-4}$, $X_{-2} = x_{-2}x_{-3}x_{-4}x_{-5}$ and $X_{-3} = x_{-3}x_{-4}x_{-5}x_{-6}$.

Proof. By means of the change of variables

$$\frac{1}{y_n} = x_n x_{n-1} x_{n-2} x_{n-3},\ n \geq -3, \tag{2.1}$$

the equation in (1.1) becomes

$$y_n = a y_{n-3} + b,\ n \in \mathbb{N}_0. \tag{2.2}$$

From (2.2), we can write

$$y_{3n+i} = a y_{3(n-1)+i} + b,\ n \in \mathbb{N}_0,\ i = 0,1,2. \tag{2.3}$$

By employing the well-known formula for the solution of linear difference equations, we get

$$y_{3m+i} = y_{i-3} a^{m+1} + b \frac{1-a^{m+1}}{1-a},\ m \in \mathbb{N}_0, \tag{2.4}$$

If $a \neq 1$, and

$$y_{3m+i} = y_{i-3} + (m+1)b,\ m \in \mathbb{N}_0, \tag{2.5}$$

If $a = 1$. From the change of variables (2.1), we get the equation

$$\begin{aligned} x_{4n+i_1} &= \frac{1}{y_{4n+i_1} x_{4n+i_1-1} x_{4n+i_1-2} x_{4n+i_1-3}} \\ &= \frac{y_{4n+i_1-1}}{y_{4n+i_1}} x_{4(n-1)+i_1} \\ &= \frac{y_{4n+i_1-1}}{y_{4n+i_1}} \frac{y_{4n+i_1-5}}{y_{4n+i_1-4}} \frac{y_{4n+i_1-9}}{y_{4n+i_1-8}} x_{4(n-3)+i_1}, \end{aligned} \tag{2.6}$$

$i_1 \in \{0,1,2,3\}$, from which it follows that

$$x_{12m+3j+i}=x_{3j+i-12}\prod_{s=0}^{m}\frac{y_{12s+3j+i-1}}{y_{12s+3j+i}}\frac{y_{12s+3j+i-5}}{y_{12s+3j+i-4}}\frac{y_{12s+3j+i-9}}{y_{12s+3j+i-8}}, \tag{2.7}$$

for each $j\in\{2,3,4,5\}$ and $i\in\{0,1,2\}$ and every $m\geq -1.$ Using (2.4) and (2.5) in (2.7) we can write the solution of Eq. (1.1) as follows

$$x_{12m+3j}=x_{3j-12}\prod_{s=0}^{m}\frac{bX_{-1}+(1-a-bX_{-1})a^{4s+j}}{bX_{-3}+(1-a-bX_{-3})a^{4s+j+1}}\frac{bX_{-2}+(1-a-bX_{-2})a^{4s+j-1}}{bX_{-1}+(1-a-bX_{-1})a^{4s+j-1}}\frac{bX_{-3}+(1-a-bX_{-3})a^{4s+j-2}}{bX_{-2}+(1-a-bX_{-2})a^{4s+j-2}},$$

$$x_{12m+3j+1}=x_{3j-11}\prod_{s=0}^{m}\frac{bX_{-3}+(1-a-bX_{-3})a^{4s+j+1}}{bX_{-2}+(1-a-bX_{-2})a^{4s+j+1}}\frac{bX_{-1}+(1-a-bX_{-1})a^{4s+j-1}}{bX_{-3}+(1-a-bX_{-3})a^{4s+j}}\frac{bX_{-2}+(1-a-bX_{-2})a^{4s+j-2}}{bX_{-1}+(1-a-bX_{-1})a^{4s+j-2}},$$

$$x_{12m+3j+2}=x_{3j-10}\prod_{s=0}^{m}\frac{bX_{-2}+(1-a-bX_{-2})a^{4s+j+1}}{bX_{-1}+(1-a-bX_{-1})a^{4s+j+1}}\frac{bX_{-3}+(1-a-bX_{-3})a^{4s+j}}{bX_{-2}+(1-a-bX_{-2})a^{4s+j}}\frac{bX_{-1}+(1-a-bX_{-1})a^{4s+j-2}}{bX_{-3}+(1-a-bX_{-3})a^{4s+j-1}},$$

If $a\neq 1,$ and

$$x_{12m+3j}=x_{3j-12}\prod_{s=0}^{m}\frac{1+(4s+j)bX_{-1}}{1+(4s+j+1)bX_{-3}}\frac{1+(4s+j-1)bX_{-2}}{1+(4s+j-1)bX_{-1}}\frac{1+(4s+j-2)bX_{-3}}{1+(4s+j-2)bX_{-2}},$$

$$x_{12m+3j+1}=x_{3j-11}\prod_{s=0}^{m}\frac{1+(4s+j+1)bX_{-3}}{1+(4s+j+1)bX_{-2}}\frac{1+(4s+j-1)bX_{-1}}{1+(4s+j)bX_{-3}}\frac{1+(4s+j-2)bX_{-2}}{1+(4s+j-2)bX_{-1}},$$

$$x_{12m+3j+2}=x_{3j-10}\prod_{s=0}^{m}\frac{1+(4s+j+1)bX_{-2}}{1+(4s+j+1)bX_{-1}}\frac{1+(4s+j)bX_{-3}}{1+(4s+j)bX_{-2}}\frac{1+(4s+j-2)bX_{-1}}{1+(4s+j-1)bX_{-3}},$$

If $a=1,$ where $X_{-1}=x_{-1}x_{-2}x_{-3}x_{-4},$ $X_{-2}=x_{-2}x_{-3}x_{-4}x_{-5}$ and $X_{-3}=x_{-3}x_{-4}x_{-5}x_{-6}.$ ■

The following results follow from above Theorem.

Corallary 2.2. Suppose that $a=-1, b\neq 0, x_{-i_1}\neq 0, i_1\in\{1,2,...,6\},$ and $(x_n)_{n\geq -6}$ is a well-defined solution of Eq. (1.1). Then, for $m\geq -1$ and $j\in\{2,3,4,5\},$ we have

$$x_{12m+3j}=x_{3j-12}\prod_{s=0}^{m}\frac{bX_{-1}+(2-bX_{-1})(-1)^{4s+j}}{bX_{-3}+(2-bX_{-3})(-1)^{4s+j+1}}\frac{bX_{-2}+(2-bX_{-2})(-1)^{4s+j-1}}{bX_{-1}+(2-bX_{-1})(-1)^{4s+j-1}}\frac{bX_{-3}+(2-bX_{-3})(-1)^{4s+j-2}}{bX_{-2}+(2-bX_{-2})(-1)^{4s+j-2}},$$

$$x_{12m+3j+1}=x_{3j-11}\prod_{s=0}^{m}\frac{bX_{-3}+(2-bX_{-3})(-1)^{4s+j+1}}{bX_{-2}+(2-bX_{-2})(-1)^{4s+j+1}}\frac{bX_{-1}+(2-bX_{-1})(-1)^{4s+j-1}}{bX_{-3}+(2-bX_{-3})(-1)^{4s+j}}\frac{bX_{-2}+(2-bX_{-2})(-1)^{4s+j-2}}{bX_{-1}+(2-bX_{-1})(-1)^{4s+j-2}},$$

$$x_{12m+3j+2}=x_{3j-10}\prod_{s=0}^{m}\frac{bX_{-2}+(2-bX_{-2})(-1)^{4s+j+1}}{bX_{-1}+(2-bX_{-1})(-1)^{4s+j+1}}\frac{bX_{-3}+(2-bX_{-3})(-1)^{4s+j}}{bX_{-2}+(2-bX_{-2})(-1)^{4s+j}}\frac{bX_{-1}+(2-bX_{-1})(-1)^{4s+j-2}}{bX_{-3}+(2-bX_{-3})(-1)^{4s+j-1}},$$

where $X_{-1}=x_{-1}x_{-2}x_{-3}x_{-4}$, $X_{-2}=x_{-2}x_{-3}x_{-4}x_{-5}$ and $X_{-3}=x_{-3}x_{-4}x_{-5}x_{-6}$. ■

Corallary 2.3. Let $(x_n)_{n\geq -6}$ be a well defined-solution of Eq. (1.1) with $a\neq 0$ and $b=0$. Then, for $m\geq 1$ and $k=6,7,...,17$, we get

$$x_{12m+k}=\frac{x_{k-12}}{a^{m+1}}.$$

Teorem 2.4. *Suppose that* $a\neq -1, b\neq 0, x_{-i_1}\neq 0, i_1=\overline{1,6}$, *and* $(x_n)_{n\geq -6}$ *is a well defined solution of* Eq. (1.1). *Then, the next statements hold.*

a) *If* $|a|>1$, $X_{-1}\neq \frac{1-a}{b}$, $X_{-2}\neq \frac{1-a}{b}$ *and* $X_{-3}\neq \frac{1-a}{b}$, *then* $x_m\to 0$ *as* $m\to\infty$.

b) *If* $|a|>1$, $X_{-1}=\frac{1-a}{b}$, $X_{-2}=\frac{1-a}{b}$ *and* $X_{-3}\neq \frac{1-a}{b}$, *then* $x_{12m+3j}\to 0$, $|x_{12m+j+1}|\to\infty$ *and* $|x_{12m+3j+2}|\to\infty$ *as* $m\to\infty$.

c) *If* $|a|>1$, $X_{-1}=\frac{1-a}{b}$, $X_{-2}\neq \frac{1-a}{b}$ *and* $X_{-3}=\frac{1-a}{b}$, *then* $|x_{12m+3j}|\to\infty$, $x_{12m+3j+1}\to 0$ *and* $|x_{12m+3j+2}|\to\infty$ *as* $m\to\infty$.

d) *If* $|a|>1$, $X_{-1}\neq \frac{1-a}{b}$, $X_{-2}=\frac{1-a}{b}$ *and* $X_{-3}=\frac{1-a}{b}$, *then* $|x_{12m+3j}|\to\infty$, $|x_{12m+3j+1}|\to\infty$ *and* $x_{12m+3j+2}\to 0$ *as* $m\to\infty$.

e) *If* $|a|>1$, $X_{-1}=\frac{1-a}{b}$, $X_{-2}\neq \frac{1-a}{b}$ *and* $X_{-3}\neq \frac{1-a}{b}$, *then* $x_{12m+3j}\to 0$, $x_{12m+j+1}\to 0$ *and* $|x_{12m+3j+2}|\to\infty$ *as* $m\to\infty$.

f) *If* $|a|>1$, $X_{-1}\neq \frac{1-a}{b}$, $X_{-2}=\frac{1-a}{b}$ *and* $X_{-3}\neq \frac{1-a}{b}$, *then* $x_{12m+3j}\to 0$, $|x_{12m+j+1}|\to\infty$ *and* $x_{12m+3j+2}\to 0$ *as* $m\to\infty$.

g) *If* $|a|>1$, $X_{-1}\neq \frac{1-a}{b}$, $X_{-2}\neq \frac{1-a}{b}$ *and* $X_{-3}=\frac{1-a}{b}$, , *then* $|x_{12m+3j}|\to\infty$, $x_{12m+3j+1}\to 0$ *and* $x_{12m+3j+2}\to 0$ *as* $m\to\infty$.

h) *If* $|a|<1$, *then the sequences* $(x_{12m+i})_{m\in\mathbb{N}_0}$ *converge for every* $i=\overline{0,11}$.

i) *If* $X_{-1}=X_{-2}=X_{-3}=\frac{1-a}{b}$, *then* $x_{12m+j}=x_{j-12}$, $m\in\mathbb{N}_0$, $j=\overline{6,17}$.

j) *If* $a=0$, *then the sequence* $(x_n)_{n\geq -6}$ *is 4-periodic.*

k) *If* $a=1$, *then* $x_n\to 0$ *as* $n\to\infty$,

where $X_{-1}=x_{-1}x_{-2}x_{-3}x_{-4}$, $X_{-2}=x_{-2}x_{-3}x_{-4}x_{-5}$ and $X_{-3}=x_{-3}x_{-4}x_{-5}x_{-6}$.

Proof. We will present the proof for each cases seperately. Let

$$w_m^{3j}=\frac{bX_{-1}+(1-a-bX_{-1})a^{4m+j}}{bX_{-3}+(1-a-bX_{-3})a^{4m+j+1}}\frac{bX_{-2}+(1-a-bX_{-2})a^{4m+j-1}}{bX_{-1}+(1-a-bX_{-1})a^{4m+j-1}}\frac{bX_{-3}+(1-a-bX_{-3})a^{4m+j-2}}{bX_{-2}+(1-a-bX_{-2})a^{4m+j-2}}, \quad (2.8)$$

$$w_m^{3j+1}=\frac{bX_{-3}+(1-a-bX_{-3})a^{4m+j+1}}{bX_{-2}+(1-a-bX_{-2})a^{4m+j+1}}\frac{bX_{-1}+(1-a-bX_{-1})a^{4m+j-1}}{bX_{-3}+(1-a-bX_{-3})a^{4m+j}}\frac{bX_{-2}+(1-a-bX_{-2})a^{4m+j-2}}{bX_{-1}+(1-a-bX_{-1})a^{4m+j-2}} \quad (2.9)$$

and

$$w_m^{3j+2}=\frac{bX_{-2}+\left(1-a-bX_{-2}\right)a^{4m+j+1}}{bX_{-1}+\left(1-a-bX_{-1}\right)a^{4m+j+1}}\frac{bX_{-3}+\left(1-a-bX_{-3}\right)a^{4m+j}}{bX_{-2}+\left(1-a-bX_{-2}\right)a^{4m+j}}\frac{bX_{-1}+\left(1-a-bX_{-1}\right)a^{4m+j-2}}{bX_{-3}+\left(1-a-bX_{-3}\right)a^{4m+j-1}}, \quad (2.10)$$

for every $j\in\{2,3,4,5\}$ and $m\in\mathbb{N}_0$, where $X_{-1}=x_{-1}x_{-2}x_{-3}x_{-4}$, $X_{-2}=x_{-2}x_{-3}x_{-4}x_{-5}$ and $X_{-3}=x_{-3}x_{-4}x_{-5}x_{-6}$.

(a): When $|a|>1$, $X_{-1}\neq\frac{1-a}{b}$, $X_{-2}\neq\frac{1-a}{b}$ *and* $X_{-3}\neq\frac{1-a}{b}$, we have

$$\lim_{m\to\infty}w_m^{3j}=\frac{1}{a},\ \lim_{m\to\infty}w_m^{3j+1}=\frac{1}{a},\ \lim_{m\to\infty}w_m^{3j+2}=\frac{1}{a} \quad (2.11)$$

for every $j\in\{2,3,4,5\}$, from which along with Theorem 2.1, the result can be seen easily.

(b)-(d): We will exclusively prove (b) since the others can be proven similarly. When $|a|>1$, $X_{-1}=\frac{1-a}{b}$, $X_{-2}=\frac{1-a}{b}$ *and* $X_{-3}\neq\frac{1-a}{b}$, we get

$$\lim_{m\to\infty}w_m^{3j}=\frac{1}{a^3},\ \lim_{m\to\infty}w_m^{3j+1}=a,\ \lim_{m\to\infty}w_m^{3j+2}=a \quad (2.12)$$

for every $j\in\{2,3,4,5\}$, from which along with Theorem 2.1, the results in (b) can be obtained easily.

(e)-(g): We will solely prove (e) since the others can be thought in the same manner with it. When $|a|>1$, $X_{-1}=\frac{1-a}{b}$, $X_{-2}\neq\frac{1-a}{b}$ *and* $X_{-3}\neq\frac{1-a}{b}$, we have

$$\lim_{m\to\infty}w_m^{3j}=\frac{1}{a^2},\ \lim_{m\to\infty}w_m^{3j+1}=\frac{1}{a^2},\ \lim_{m\to\infty}w_m^{3j+2}=a^2, \quad (2.13)$$

for every $j\in\{2,3,4,5\}$, from which along with Theorem 2.1, the results in (e) follow easily.

(h): Employing the Taylor expansion for $(1+x)^{-1}$ on the interval $(-\varepsilon,\varepsilon)$, where $\varepsilon>0$, we get, for sufficiently large *m*,

$$\begin{aligned}
w_m^{3j}&=\frac{bX_{-1}+\left(1-a-bX_{-1}\right)a^{4m+j}}{bX_{-3}+\left(1-a-bX_{-3}\right)a^{4m+j+1}}\frac{bX_{-2}+\left(1-a-bX_{-2}\right)a^{4m+j-1}}{bX_{-1}+\left(1-a-bX_{-1}\right)a^{4m+j-1}}\frac{bX_{-3}+\left(1-a-bX_{-3}\right)a^{4m+j-2}}{bX_{-2}+\left(1-a-bX_{-2}\right)a^{4m+j-2}}\\
&=1+\frac{(1-a-bX_{-1})}{bX_{-1}}\left(1-\frac{1}{a}\right)a^{4m+j}+\frac{(1-a-bX_{-2})}{bX_{-2}}\left(\frac{a-1}{a^2}\right)a^{4m+j}\\
&+\frac{(1-a-bX_{-3})}{bX_{-3}}\left(\frac{1-a^3}{a^2}\right)a^{4m+j}+\frac{(1-a-bX_{-1})}{bX_{-1}}\left(1-\frac{1}{a}\right)\frac{(1-a-bX_{-2})}{bX_{-2}}\left(\frac{a-1}{a^2}\right)a^{8m+2j}\\
&+\frac{(1-a-bX_{-1})}{bX_{-1}}\left(1-\frac{1}{a}\right)\frac{(1-a-bX_{-3})}{bX_{-3}}\left(\frac{1-a^3}{a^2}\right)a^{8m+2j}\\
&+\frac{(1-a-bX_{-2})}{bX_{-2}}\left(\frac{a-1}{a^2}\right)\frac{(1-a-bX_{-3})}{bX_{-3}}\left(\frac{1-a^3}{a^2}\right)a^{8m+2j}+\mathcal{O}\left(a^{4m}\right),
\end{aligned} \quad (2.14)$$

$$
\begin{aligned}
w_m^{3j+1} &= \frac{bX_{-3}+(1-a-bX_{-3})a^{4m+j+1}}{bX_{-2}+(1-a-bX_{-2})a^{4m+j+1}}\frac{bX_{-1}+(1-a-bX_{-1})a^{4m+j-1}}{bX_{-3}+(1-a-bX_{-3})a^{4m+j}}\frac{bX_{-2}+(1-a-bX_{-2})a^{4m+j-2}}{bX_{-1}+(1-a-bX_{-1})a^{4m+j-2}}\\
&=1+\frac{(1-a-bX_{-3})}{bX_{-3}}(a-1)a^{4m+j}+\frac{(1-a-bX_{-1})}{bX_{-1}}\left(\frac{1}{a}-\frac{1}{a^2}\right)a^{4m+j}\\
&+\frac{(1-a-bX_{-2})}{bX_{-2}}\left(\frac{1}{a^2}-a\right)a^{4m+j}+\frac{(1-a-bX_{-3})}{bX_{-3}}(a-1)\frac{(1-a-bX_{-1})}{bX_{-1}}\left(\frac{1}{a}-\frac{1}{a^2}\right)a^{8m+2j}\\
&+\frac{(1-a-bX_{-3})}{bX_{-3}}(a-1)\frac{(1-a-bX_{-2})}{bX_{-2}}\left(\frac{1}{a^2}-a\right)a^{8m+2j}\\
&+\frac{(1-a-bX_{-1})}{bX_{-1}}\left(\frac{1}{a}-\frac{1}{a^2}\right)\frac{(1-a-bX_{-2})}{bX_{-2}}\left(\frac{1}{a^2}-a\right)a^{8m+2j}+\mathcal{O}(a^{4m}),
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
w_m^{3j+2} &= \frac{bX_{-2}+(1-a-bX_{-2})a^{4m+j+1}}{bX_{-1}+(1-a-bX_{-1})a^{4m+j+1}}\frac{bX_{-3}+(1-a-bX_{-3})a^{4m+j}}{bX_{-2}+(1-a-bX_{-2})a^{4m+j}}\frac{bX_{-1}+(1-a-bX_{-1})a^{4m+j-2}}{bX_{-3}+(1-a-bX_{-3})a^{4m+j-1}}\\
&=1+\frac{(1-a-bX_{-2})}{bX_{-2}}(a-1)a^{4m+j}+\frac{(1-a-bX_{-3})}{bX_{-3}}\left(1-\frac{1}{a}\right)a^{4m+j}\\
&+\frac{(1-a-bX_{-1})}{bX_{-1}}\left(\frac{1}{a^2}-a\right)a^{4m+j}+\frac{(1-a-bX_{-2})}{bX_{-2}}(a-1)\frac{(1-a-bX_{-3})}{bX_{-3}}\left(1-\frac{1}{a}\right)a^{8m+2j}\\
&+\frac{(1-a-bX_{-3})}{bX_{-3}}\left(1-\frac{1}{a}\right)\frac{(1-a-bX_{-1})}{bX_{-1}}\left(\frac{1}{a^2}-a\right)a^{8m+2j}\\
&+\frac{(1-a-bX_{-2})}{bX_{-2}}(a-1)\frac{(1-a-bX_{-1})}{bX_{-1}}\left(\frac{1}{a^2}-a\right)a^{8m+2j}+\mathcal{O}(a^{4m}).
\end{aligned}
\tag{2.16}
$$

The results follow from (2.14) - (2.16) and the condition $|a|<1$.

(i): This result can be seen easily from the assumption $X_{-1}=X_{-2}=X_{-3}=\frac{1-a}{b}$ and some simple calculation.

(j): If $a=0$, from (1.1) we get

$$
x_n=\frac{1}{bx_{n-1}x_{n-2}x_{n-3}}=\frac{1}{b\frac{1}{bx_{n-2}x_{n-3}x_{n-4}}x_{n-2}x_{n-3}}=x_{n-4},\ n\in\mathbb{N}_0, \tag{2.17}
$$

from which the result easily follows.

(k): For each $j\in\{2,3,4,5\}$ and suffciently large *m*, we obtain

$$
\begin{aligned}
x_{12m+3j} &= x_{3j-12}\prod_{s=0}^{m}\frac{1+\ 4s+j\ bX_{-1}}{1+\ 4s+j+1\ bX_{-3}}\frac{1+\ 4s+j-1\ bX_{-2}}{1+\ 4s+j-1\ bX_{-1}}\frac{1+\ 4s+j-2\ bX_{-3}}{1+\ 4s+j-2\ bX_{-2}}\\
&= x_{3j-12}C_1(m_0)\prod_{s=m_0+1}^{m}\frac{1+\frac{1+jbX_{-1}}{4sbX_{-1}}}{1+\frac{1+(j+1)bX_{-3}}{4sbX_{-3}}}\frac{1+\frac{1+(j-1)bX_{-2}}{4sbX_{-2}}}{1+\frac{1+(j-1)bX_{-1}}{4sbX_{-1}}}\frac{\left(1+\frac{1+(j-2)bX_{-3}}{4sbX_{-3}}\right)}{\left(1+\frac{1+(j-2)bX_{-2}}{4sbX_{-2}}\right)}\\
&= x_{3j-12}C_1(m_0)\prod_{s=m_0+1}^{m}\left(1+\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\left(1+\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\left(1-\frac{3}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right) \qquad (2.18)\\
&= x_{3j-12}C_1(m_0)e^{\sum_{s=m_0+1}^{m}\ln\left(1-\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)}\\
&= x_{3j-12}C_1(m_0)e^{\left(-\frac{1}{4}\sum_{s=m_0+1}^{m}\left(\frac{1}{s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\right)},
\end{aligned}
$$

$$
\begin{aligned}
x_{12m+3j+1} &= x_{3j-11}\prod_{s=0}^{m}\frac{1+\ 4s+j+1\ bX_{-3}}{1+\ 4s+j+1\ bX_{-2}}\frac{1+\ 4s+j-1\ bX_{-1}}{1+\ 4s+j\ bX_{-3}}\frac{1+\ 4s+j-2\ bX_{-2}}{1+\ 4s+j-2\ bX_{-1}}\\
&= x_{3j-11}C_2(m_0)\prod_{s=m_0+1}^{m}\frac{1+\frac{1+(j+1)bX_{-3}}{4sbX_{-3}}}{1+\frac{1+(j+1)bX_{-2}}{4sbX_{-2}}}\frac{1+\frac{1+(j-1)bX_{-1}}{4sbX_{-1}}}{1+\frac{1+jbX_{-3}}{4sbX_{-3}}}\frac{\left(1+\frac{1+(j-2)bX_{-2}}{4sbX_{-2}}\right)}{\left(1+\frac{1+(j-2)bX_{-1}}{4sbX_{-1}}\right)}\\
&= x_{3j-11}C_2(m_0)\prod_{s=m_0+1}^{m}\left(1+\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\left(1+\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\left(1-\frac{3}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right) \qquad (2.19)\\
&= x_{3j-11}C_2(m_0)e^{\sum_{s=m_0+1}^{m}\ln\left(1-\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)}\\
&= x_{3j-11}C_2(m_0)e^{\left(-\frac{1}{4}\sum_{s=m_0+1}^{m}\left(\frac{1}{s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\right)},
\end{aligned}
$$

$$
\begin{aligned}
x_{12m+3j+2} &= x_{3j-10}\prod_{s=0}^{m}\frac{1+\ 4s+j+1\ bX_{-2}}{1+\ 4s+j+1\ bX_{-1}}\frac{1+\ 4s+j\ bX_{-3}}{1+\ 4s+j\ bX_{-2}}\frac{1+\ 4s+j-2\ bX_{-1}}{1+\ 4s+j-1\ bX_{-3}}\\
&= x_{3j-10}C_3(m_0)\prod_{s=m_0+1}^{m}\frac{1+\frac{1+(j+1)bX_{-2}}{4sbX_{-2}}}{1+\frac{1+(j+1)bX_{-1}}{4sbX_{-1}}}\frac{1+\frac{1+jbX_{-3}}{4sbX_{-3}}}{1+\frac{1+jbX_{-2}}{4sbX_{-2}}}\frac{1+\frac{1+(j-2)bX_{-1}}{4sbX_{-1}}}{\left(1+\frac{1+(j-1)bX_{-3}}{4sbX_{-3}}\right)}\\
&= x_{3j-10}C_3(m_0)\prod_{s=m_0+1}^{m}\left(1+\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\left(1+\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\left(1-\frac{3}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right) \qquad (2.20)\\
&= x_{3j-10}C_3(m_0)e^{\sum_{s=m_0+1}^{m}\ln\left(1-\frac{1}{4s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)}\\
&= x_{3j-10}C_3(m_0)e^{\left(-\frac{1}{4}\sum_{s=m_0+1}^{m}\left(\frac{1}{s}+\mathcal{O}\left(\frac{1}{s^2}\right)\right)\right)}.
\end{aligned}
$$

From (2.18) -(2.20) and employing the fact that $\sum_{j_1=1}^{m}\left(\frac{1}{j_1}\right)\to\infty$ as $m\to\infty$, then the statement can be seen easily.■

The following theorem gives the behavior of well-defined solution of Eq. (1.1) in the case $a=-1,\ b\neq 0$.

Theorem 2.5. *Suppose that* $a=-1, b\neq 0,\ x_{-i_1}\neq 0,\ i_1=1,2,...,6,$ *and* $(x_n)_{n\geq -6}$ *is a well-defined solution of Eq. (1.1). Let* $N_i=bx_{-i-1}x_{-i-2}x_{-i-3}x_{-i-4}-1,$ *for* $i=0,1,2$ *and* $t=1,2.$ *Then the next statements hold.*

a) *If* $X_{-1}=X_{-2}=X_{-3}=\frac{2}{b},$ *, then* $(x_n)_{n\geq -6}$ *is twelve-periodic.*

b) *If* $X_{-2}=X_{-3}=\frac{2}{b}$ *and* $|N_0|<1,$ *then* $|x_{12m+6t+2i}|\to\infty$ *and* $x_{12m+6t+2i+1}\to 0,$ *as* $m\to\infty.$

c) *If* $X_{-2}=X_{-3}=\frac{2}{b}$ *and* $|N_0|>1,$ *then* $x_{12m+6t+2i}\to 0$ *and* $|x_{12m+6t+2i+1}|\to\infty,$ *as* $m\to\infty.$

d) *If* $X_{-1}=X_{-3}=\frac{2}{b}$ *and* $|N_1|<1,$ *then* $x_{12m+6t+2i}\to 0$ *and* $|x_{12m+6t+2i+1}|\to\infty,$ *as* $m\to\infty.$

e) *If* $X_{-1}=X_{-3}=\frac{2}{b}$ *and* $|N_1|>1,$ *then* $|x_{12m+6t+2i}|\to\infty$ *and* $x_{12m+6t+2i+1}\to 0,$ *as* $m\to\infty.$

f) *If* $X_{-1}=X_{-2}=\frac{2}{b}$ *and* $|N_2|<1,$ *then* $|x_{12m+6t+2i}|\to\infty$ *and* $x_{12m+6t+2i+1}\to 0,$ *as* $m\to\infty.$

g) *If* $X_{-1}=X_{-2}=\frac{2}{b}$ *and* $|N_2|>1,$ *then* $x_{12m+6t+2i}\to 0$ *and* $|x_{12m+6t+2i+1}|\to\infty,$ *as* $m\to\infty.$

h) *If* $X_{-1}=\frac{2}{b}, X_{-2}\neq\frac{2}{b}, X_{-3}\neq\frac{2}{b}, X_{-2}\neq\frac{1}{b}, X_{-3}\neq\frac{1}{b}$ *and* $|\frac{N_1}{N_2}|<1,$ *then* $x_{12m+6t+2i}\to 0$ *and* $|x_{12m+6t+2i+1}|\to\infty,$ *as* $m\to\infty.$

i) *If* $X_{-1}=\frac{2}{b}, X_{-2}\neq\frac{2}{b}, X_{-3}\neq\frac{2}{b}, X_{-2}\neq\frac{1}{b}, X_{-3}\neq\frac{1}{b}$ *and* $|\frac{N_1}{N_2}|>1$ *, then* $|x_{12m+6t+2i}|\to\infty$ *and* $x_{12m+6t+2i+1}\to 0,$ *as* $m\to\infty.$

j) *If* $X_{-1}=\frac{2}{b}$ *and* $\frac{N_1}{N_2}=1,$ *then* $(x_n)_{n\geq -6}$ *is twelve-periodic.*

k) *If* $X_{-1}=\frac{2}{b}$ *and* $\frac{N_1}{N_2}=-1,$ *then* $(x_n)_{n\geq -6}$ *is twenty four-periodic.*

l) *If* $X_{-2}=\frac{2}{b}, X_{-1}\neq\frac{2}{b}, X_{-3}\neq\frac{2}{b}, X_{-1}\neq\frac{1}{b}, X_{-3}\neq\frac{1}{b}$ *and* $|N_0N_2|<1$ *, then* $|x_{12m+6t+2i}|\to\infty$ *and* $x_{12m+6t+2i+1}\to 0,$ *as* $m\to\infty.$

m) *If* $X_{-2}=\frac{2}{b}, X_{-1}\neq\frac{2}{b}, X_{-3}\neq\frac{2}{b}, X_{-1}\neq\frac{1}{b}, X_{-3}\neq\frac{1}{b}$ *and* $|N_0N_2|>1,$ *then* $x_{12m+6t+2i}\to 0$ *and* $|x_{12m+6t+2i+1}|\to\infty,$ *as* $m\to\infty.$

n) *If* $X_{-2}=\frac{2}{b}$ *and* $N_0N_2=1$ *, then* $(x_n)_{n\geq -6}$ *is twelve-periodic.*

o) *If* $X_{-2}=\frac{2}{b}$ *and* $N_0N_2=-1$ *, then* $(x_n)_{n\geq -6}$ *is twenty four-periodic.*

p) *If* $X_{-3}=\frac{2}{b}, X_{-1}\neq\frac{1}{b}, X_{-2}\neq\frac{1}{b}$ *and* $|\frac{N_0}{N_1}|<1,$ *then* $|x_{12m+6t+2i}|\to\infty$ *and* $x_{12m+6t+2i+1}\to 0,$ *as* $m\to\infty.$

q) *If* $X_{-3}=\frac{2}{b}, X_{-1}\neq\frac{1}{b}, X_{-2}\neq\frac{1}{b}$ *and* $|\frac{N_0}{N_1}|>1,$ *then* $x_{12m+6t+2i}\to 0$ *and* $|x_{12m+6t+2i+1}|\to\infty,$ *as* $m\to\infty.$

r) *If* $X_{-3}=\frac{2}{b}$ *and* $\frac{N_0}{N_1}=1$ *, then* $(x_n)_{n\geq -6}$ *is twelve-periodic.*

s) *If* $X_{-3}=\frac{2}{b}$ *and* $\frac{N_0}{N_1}=-1$ *, then* $(x_n)_{n\geq -6}$ *is twenty four-periodic,*

where $X_{-1}=x_{-1}x_{-2}x_{-3}x_{-4},\ X_{-2}=x_{-2}x_{-3}x_{-4}x_{-5}$ and $X_{-3}=x_{-3}x_{-4}x_{-5}x_{-6}.$

Proof.

(a): From Theorem 2.4.-(i), the result can be seen easily.

In here, we will prove the items (b)-(c) since (d)-(e) and (f)-(g) can be proved similarly.

(b)-(c):When $X_{-2} = X_{-3} = \frac{2}{b}$ and $X_{-1} \neq \frac{2}{b}$, from Corollary 2.3. we have that

$$x_{12m+3j} = x_{3j-12}\prod_{s=0}^{m}\frac{bX_{-1} + 2 - bX_{-1} (-1)^{4s+j}}{bX_{-1} + 2 - bX_{-1} (-1)^{4s+j-1}} = \frac{x_{3j-12}}{{\frac{bX_{-1} + 2 - bX_{-1} (-1)^{j-1}}{bX_{-1} + 2 - bX_{-1} (-1)^{j}}}^{m+1}}, \quad (2.21)$$

$$x_{12m+3j+1} = x_{3j-11}\prod_{s=0}^{m}\frac{bX_{-1} + 2 - bX_{-1} (-1)^{4s+j-1}}{bX_{-1} + 2 - bX_{-1} (-1)^{4s+j-2}} = \frac{x_{3j-11}}{{\frac{bX_{-1} + 2 - bX_{-1} (-1)^{j-2}}{bX_{-1} + 2 - bX_{-1} (-1)^{j-1}}}^{m+1}}, \quad (2.22)$$

$$x_{12m+3j+2} = x_{3j-10}\prod_{s=0}^{m}\frac{bX_{-1} + 2 - bX_{-1} (-1)^{4s+j-2}}{bX_{-1} + 2 - bX_{-1} (-1)^{4s+j+1}} = \frac{x_{3j-10}}{{\frac{bX_{-1} + 2 - bX_{-1} (-1)^{j+1}}{bX_{-1} + 2 - bX_{-1} (-1)^{j-2}}}^{m+1}}. \quad (2.23)$$

In here there are two cases to be considered.

- ***j* is even case**: In this case we have

$$x_{12m+6t} = \frac{x_{6t-12}}{{\frac{bX_{-1} + 2 - bX_{-1} (-1)^{2t-1}}{bX_{-1} + 2 - bX_{-1} (-1)^{2t}}}^{m+1}} = \frac{x_{6t-12}}{bX_{-1} - 1^{\,m+1}}, \quad (2.24)$$

$$x_{12m+6t+1} = \frac{x_{6t-11}}{{\frac{bX_{-1} + 2 - bX_{-1} (-1)^{2t-2}}{bX_{-1} + 2 - bX_{-1} (-1)^{2t-1}}}^{m+1}} = \frac{x_{6t-11}}{{\frac{1}{bX_{-1} - 1}}^{m+1}}, \quad (2.25)$$

$$x_{12m+6t+2} = \frac{x_{6t-10}}{{\frac{bX_{-1} + 2 - bX_{-1} (-1)^{2t+1}}{bX_{-1} + 2 - bX_{-1} (-1)^{2t-2}}}^{m+1}} = \frac{x_{6t-10}}{bX_{-1} - 1^{\,m+1}}, \quad (2.26)$$

where $j = 2t$ and $t \in \{1, 2\}$.

- ***j* is odd case**: In this case we get

$$x_{12m+6t+3} = \frac{x_{6t-9}}{{\frac{bX_{-1} + 2 - bX_{-1} (-1)^{2t}}{bX_{-1} + 2 - bX_{-1} (-1)^{2t+1}}}^{m+1}} = \frac{x_{6t-9}}{{\frac{1}{bX_{-1} - 1}}^{m+1}}, \quad (2.27)$$

$$x_{12m+6t+4} = \frac{x_{6t-8}}{{\frac{bX_{-1} + 2 - bX_{-1} (-1)^{2t-1}}{bX_{-1} + 2 - bX_{-1} (-1)^{2t}}}^{m+1}} = \frac{x_{6t-8}}{bX_{-1} - 1^{\,m+1}}, \quad (2.28)$$

$$x_{12m+6t+5} = \frac{x_{6t-7}}{{\frac{bX_{-1} + 2 - bX_{-1} (-1)^{2t+2}}{bX_{-1} + 2 - bX_{-1} (-1)^{2t-1}}}^{m+1}} = \frac{x_{6t-7}}{{\frac{1}{bX_{-1} - 1}}^{m+1}}, \quad (2.29)$$

where $j = 2t + 1$ and $t \in \{1, 2\}$.

From (2.24)-(2.29), the results can be seen easily. (h)-(k): When $X_{-1}=\frac{2}{b}$, $X_{-2}\neq\frac{1}{b}$, $X_{-3}\neq\frac{1}{b}$, $X_{-2}\neq\frac{2}{b}$, $X_{-3}\neq\frac{2}{b}$, and from Corollary 2.3., we have

$$\begin{aligned} x_{12m+3j} &= x_{3j-12}\prod_{s=0}^{m}\frac{bX_{-2}+(2-bX_{-2})(-1)^{4s+j-1}}{bX_{-3}+(2-bX_{-3})(-1)^{4s+j+1}}\frac{bX_{-3}+(2-bX_{-3})(-1)^{4s+j-2}}{bX_{-2}+(2-bX_{-2})(-1)^{4s+j-2}} \\ &= \frac{x_{3j-12}}{\left(\frac{bX_{-3}+(2-bX_{-3})(-1)^{j+1}}{bX_{-2}+(2-bX_{-2})(-1)^{j-1}}\right)^{m+1}\left(\frac{bX_{-2}+(2-bX_{-2})(-1)^{j-2}}{bX_{-3}+(2-bX_{-3})(-1)^{j-2}}\right)^{m+1}}, \end{aligned} \tag{2.30}$$

$$\begin{aligned} x_{12m+3j+1} &= x_{3j-11}\prod_{s=0}^{m}\frac{bX_{-3}+(2-bX_{-3})(-1)^{4s+j+1}}{bX_{-2}+(2-bX_{-2})(-1)^{4s+j+1}}\frac{bX_{-2}+(2-bX_{-2})(-1)^{4s+j-2}}{bX_{-3}+(2-bX_{-3})(-1)^{4s+j}} \\ &= \frac{x_{3j-11}}{\left(\frac{bX_{-2}+(2-bX_{-2})(-1)^{j+1}}{bX_{-3}+(2-bX_{-3})(-1)^{j+1}}\right)^{m+1}\left(\frac{bX_{-3}+(2-bX_{-3})(-1)^{j}}{bX_{-2}+(2-bX_{-2})(-1)^{j-2}}\right)^{m+1}}, \end{aligned} \tag{2.31}$$

$$\begin{aligned} x_{12m+3j+2} &= x_{3j-10}\prod_{s=0}^{m}\frac{bX_{-2}+(2-bX_{-2})(-1)^{4s+j+1}}{bX_{-3}+(2-bX_{-3})(-1)^{4s+j-1}}\frac{bX_{-3}+(2-bX_{-3})(-1)^{4s+j}}{bX_{-2}+(2-bX_{-2})(-1)^{4s+j}} \\ &= \frac{x_{3j-10}}{\left(\frac{bX_{-3}+(2-bX_{-3})(-1)^{j-1}}{bX_{-2}+(2-bX_{-2})(-1)^{j+1}}\right)^{m+1}\left(\frac{bX_{-2}+(2-bX_{-2})(-1)^{j}}{bX_{-3}+(2-bX_{-3})(-1)^{j}}\right)^{m+1}}. \end{aligned} \tag{2.32}$$

Similarly, in here there are two cases to be considered.

- ***j* is even case**: In this case we have

$$\begin{aligned} x_{12m+6t} &= \frac{x_{6t-12}}{\left(\frac{bX_{-3}+(2-bX_{-3})(-1)^{2t+1}}{bX_{-2}+(2-bX_{-2})(-1)^{2t-1}}\right)^{m+1}\left(\frac{bX_{-2}+(2-bX_{-2})(-1)^{2t-2}}{bX_{-3}+(2-bX_{-3})(-1)^{2t-2}}\right)^{m+1}} \\ &= \frac{x_{6t-12}}{\left(\frac{bX_{-3}-1}{bX_{-2}-1}\right)^{m+1}}, \end{aligned} \tag{2.33}$$

$$\begin{aligned} x_{12m+6t+1} &= \frac{x_{6t-11}}{\left(\frac{bX_{-2}+(2-bX_{-2})(-1)^{2t+1}}{bX_{-3}+(2-bX_{-3})(-1)^{2t+1}}\right)^{m+1}\left(\frac{bX_{-3}+(2-bX_{-3})(-1)^{2t}}{bX_{-2}+(2-bX_{-2})(-1)^{2t-2}}\right)^{m+1}} \\ &= \frac{x_{6t-11}}{\left(\frac{bX_{-2}-1}{bX_{-3}-1}\right)^{m+1}}, \end{aligned} \tag{2.34}$$

$$\begin{aligned} x_{12m+6t+2} &= \frac{x_{6t-10}}{\left(\frac{bX_{-3}+(2-bX_{-3})(-1)^{2t-1}}{bX_{-2}+(2-bX_{-2})(-1)^{2t+1}}\right)^{m+1}\left(\frac{bX_{-2}+(2-bX_{-2})(-1)^{2t}}{bX_{-3}+(2-bX_{-3})(-1)^{2t}}\right)^{m+1}} \\ &= \frac{x_{6t-10}}{\left(\frac{bX_{-3}-1}{bX_{-2}-1}\right)^{m+1}}, \end{aligned} \tag{2.35}$$

where $j=2t$ and $t\in\{1,2\}$.

- ***j* is odd case**: In this case we get

$$
\begin{aligned}
x_{12m+6t+3} &= \frac{x_{6t-9}}{\frac{bX_{-3}+\ 2-bX_{-3}\ (-1)^{2t+2}}{bX_{-2}+\ 2-bX_{-2}\ (-1)^{2t}}^{m+1}\ \frac{bX_{-2}+\ 2-bX_{-2}\ (-1)^{2t-1}}{bX_{-3}+\ 2-bX_{-3}\ (-1)^{2t-1}}^{m+1}} \\
&= \frac{x_{6t-9}}{\frac{bX_{-2}-1}{bX_{-3}-1}^{m+1}},
\end{aligned}
\tag{2.36}
$$

$$
\begin{aligned}
x_{12m+6t+4} &= \frac{x_{6t-8}}{\frac{bX_{-2}+\ 2-bX_{-2}\ (-1)^{2t+2}}{bX_{-3}+\ 2-bX_{-3}\ (-1)^{2t+2}}^{m+1}\ \frac{bX_{-3}+\ 2-bX_{-3}\ (-1)^{2t+1}}{bX_{-2}+\ 2-bX_{-2}\ (-1)^{2t-1}}^{m+1}} \\
&= \frac{x_{6t-8}}{\frac{bX_{-3}-1}{bX_{-2}-1}^{m+1}},
\end{aligned}
\tag{2.37}
$$

$$
\begin{aligned}
x_{12m+6t+5} &= \frac{x_{6t-7}}{\frac{bX_{-3}+\ 2-bX_{-3}\ (-1)^{2t}}{bX_{-2}+\ 2-bX_{-2}\ (-1)^{2t+2}}^{m+1}\ \frac{bX_{-2}+\ 2-bX_{-2}\ (-1)^{2t+1}}{bX_{-3}+\ 2-bX_{-3}\ (-1)^{2t+1}}^{m+1}} \\
&= \frac{x_{6t-7}}{\frac{bX_{-2}-1}{bX_{-3}-1}^{m+1}},
\end{aligned}
\tag{2.38}
$$

where $j=2t+1$ and $t\in\{1,2\}$.

From (2.33)-(2.38), the results can be obtained easily.

The proofs of (l)-(o) and (p)-(s) can be proved similar to the proof of (k)-(l). So we will omit them here.■

Finally we investigate the asymptotic behavior of solution of Eq. (1.1) when $a\neq 0,\ b=0,$ by employing the following formulae, for the case $a\neq 1,\ i\in\{0,1,2\}$

$$x_{12m+3j+i}=x_{3j+i-12}\prod_{s=0}^{m}\frac{1}{a},\ m\in\mathbb{N}_0, \tag{2.39}$$

while for $a=1$,

$$x_{12m+3j+i}=x_{3j+i-12},\ m\in\mathbb{N}_0. \tag{2.40}$$

By using above formulae, we give the following theorem. Proof of the theorem can be seen easily from (2.39)-(2.40).

Theorem 2.6. *Let* $a\neq 0,\ b=0$ *and* $x_n\ _{n\geq -6}$ *be a well defined solution of* Eq. (1.1). *Then, the next statements hold.*

a) If $|a|>1,$ *then* $x_m\to 0$ *as* $m\to\infty.$

b) If $|a|<1,$ *, then* $x_m\to\infty$ *as* $m\to\infty.$

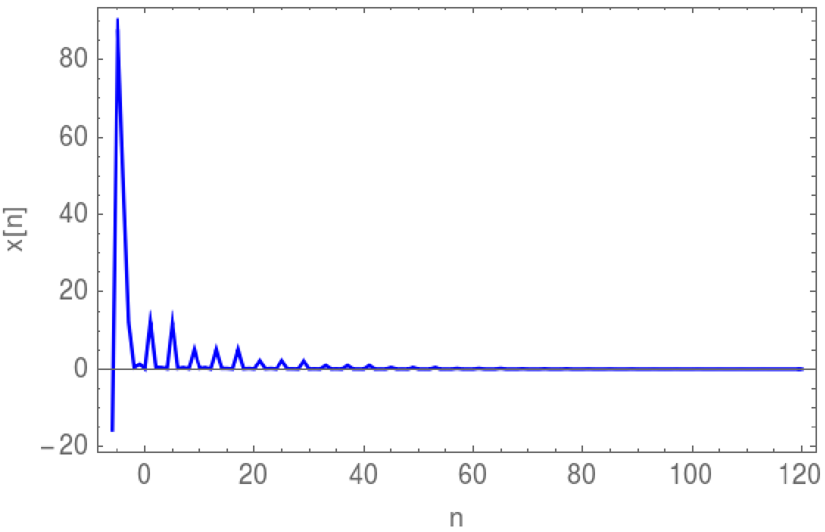
c) If $a=1,$ *then* $x_n\ _{n\geq -6}$ *is twelve-periodic.*

d) If $a=-1,$ *then* $x_n\ _{n\geq -6}$ *is twenty four-periodic.*

3. NUMERICAL EXAMPLES

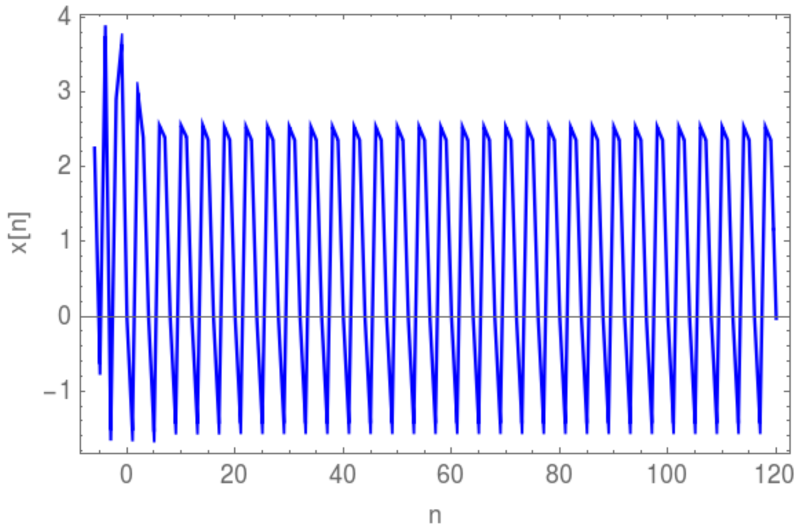
For confirming the results of this paper, we give numerical examples for the solution of Eq. (1.1) regard to the different values of a and b.

Example 3.1. *Consider the equation* (1.1) *with the initial values* $x_{-6} = -15.71$, $x_{-5} = 87.76$, $x_{-4} = 48.97$, $x_{-3} = 12.23$, $x_{-2} = -0.45$, $x_{-1} = 97.01$. *Moreover, we take the parameters* $a = 2.27$, $b = 1.34$, *i.e.,*



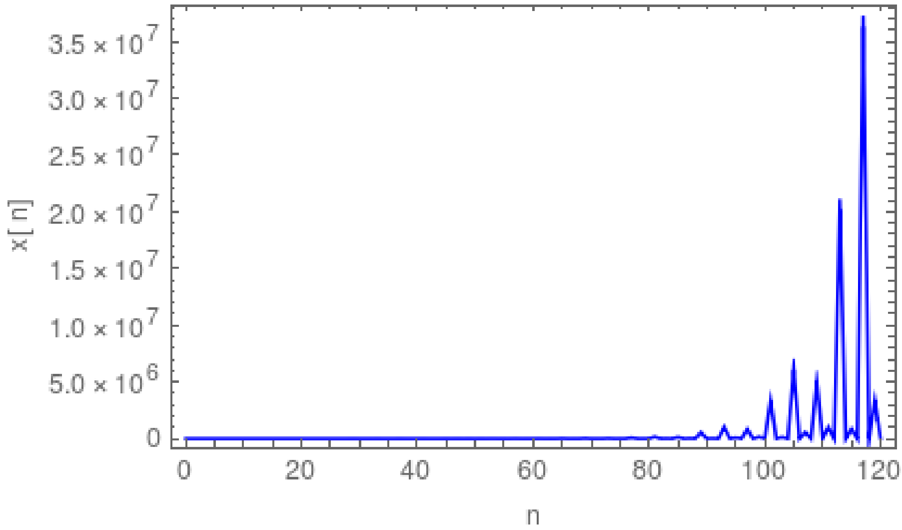
In this case the condition (a) in Theorem 2.4 is satisfied. Hence, the solution of Eq. (1.1) *tend to zero.*

Example 3.2. *Consider the equation* (1.1) *with the initial values* $x_{-6} = 2.25$, $x_{-5} = -0.64$, , $x_{-4} = 3.75$, $x_{-3} = -1.52$, $x_{-2} = 2.91$, $x_{-1} = 3.64$. *Moreover, we take the parameters* $a = 0.5$, $b = 2$, *i.e.,*



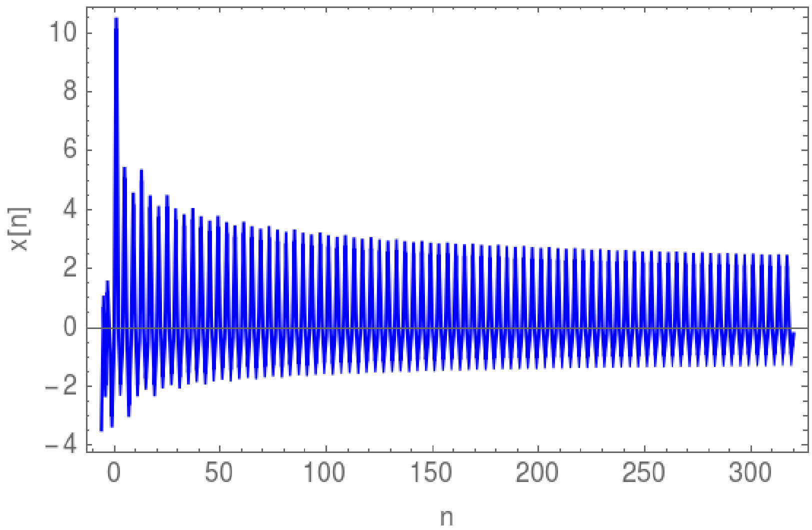
In this case the condition (h) in Theorem 2.4 is satisfied. Thus the sequence $(x_{12m+i})_{m \in \mathbb{N}_0}$ *converge for every* $i = \overline{0,11}$.

Example 3.3. *Consider the equation* (1.1) *with the initial values* $x_{-6}=0.8,\ x_{-5}=0.5,\ x_{-4}=10,\ x_{-3}=0.1,\ x_{-2}=1,\ x_{-1}=2.$ *Moreover, we take the parameters* $a=-1,\ b=5$*, i.e.,*



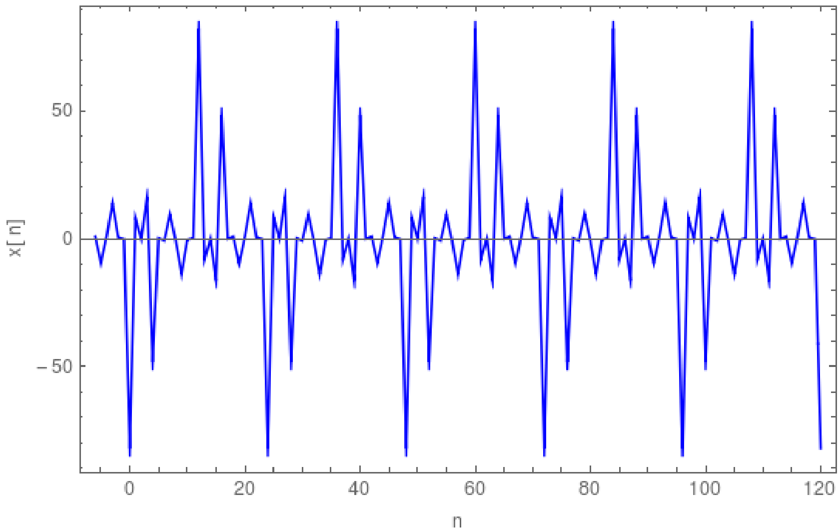
In this case the condition (q) in Theorem 2.5 is satisfied. Thus some of subsequence of the x_n *sequence converge to zero and simultaneously the absolute value of the others diverge to infinity.*

Example 3.4. *Consider the equation* (1.1) *with the initial values* $x_{-6}=-3.47,\ x_{-5}=0.7,\ x_{-4}=-1.97,\ x_{-3}=1.2,\ x_{-2}=-0.45,\ x_{-1}=-3.01.$ *Moreover, we take the parameters* $a=1,\ b=-1.5,$ *i.e.,*



In this case the condition (k) in Theorem 2.4 is satisfied. Thus, the solution of Eq. (1.1) *tend to zero.*

Example 3.5. *Consider the equation* (1.1) *with the initial values* $x_{-6}=0.85,\ x_{-5}=-9.68,$, $x_{-4}=1,\ x_{-3}=14,\ x_{-2}=0.5,\ x_{-1}=-0.2.$ *Moreover, we take the parameters* $a=-1, b=0,$ *i.e.,*



In this case the condition (d) in Theorem 2.6 is satisfied. Hence, the solution of Eq. (1.1) *has a periodic solution with period twenty-four.*

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