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ON DEVELOPABLE RULED SURFACE OF THE PRINCIPAL-DIRECTION CURVE

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Abstract. This study is devoted to improve the theory of the developable ruled surfaces in terms of principal-direction curves of any spatial curve in three-dimensional Euclidean space. We obtain the new representation for developable surfaces by slant helices and the useful elements such as pitch, angle of pitch and dral with the help of a new frame $\{N, C, W\}$. Furthermore, the investigation is observed under some special cases in terms of the director vector of surface.

1. Introduction

In differential geometry, one of the most important equipment, surfaces have the major positions and concepts in many disciplines such as physics, engineering, computer graphics, etc. One of the significant family of surfaces is called ruled surface which is investigated by G. Monge. Any ruled surface occurs as a result of the continuously movement of a straight line along any curve, which are called the director and the base curve of the surface, respectively. Moreover, ruled surfaces especially have an effective application area in kinematics, CAD, CAGD, robotics, architecture and in other many fields.

Izumiya and Takeuchi [13], Hacısalihoğlu [6] obtained some characterizations for the ruled surfaces, also kinematic approach of them being given in [2, 7, 8, 11, 12]. The viewpoint of these studies focuses on the relationship between ruled surfaces and helical curves in terms of the Frenet frame of three-dimension Euclidean space. On the other hand, if the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface, otherwise named as skew surfaces. Besides what shown in [5], if a developable surface lies in three-dimensional Euclidean space, then it is necessarily ruled but the converse is not always true. The answer of this problem is researched when the base curve is a helix curve which has the property tangent line at any point makes a constant angle with a fixed line.

In this study, we give a new characterization for the developable ruled surface which is composed of integral curve of any space curve called a principal-direction

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curve in terms of its Frenet frame $\{N, C, W\}$ with the director vector X in three-dimensional Euclidean space. Hence the developable ruled surface is also analyzed for the special cases of the director vector X when it lies in main places. With this theory, we obtain a new approach in terms of the special curves defined by the property that the principal normal makes a constant angle with a fixed direction called the slant helix. Besides, the principal elements of the ruled surface which are the pitch, angle of pitch and dral are defined in terms of the frame $\{N, C, W\}$ and the developability is represented by them. Moreover, as a result of this theory we obtain the relation which is between the elements of surface and slant helix. At the end of the study, the theory is supported by an example.

2. Preliminaries

We now recall some basic concepts and formulas of the ruled surfaces and slant helix in Euclidean 3−space. We also analyze a principal-direction curve with its Frenet frame $\{N, C, W\}$ for giving some characterizations of the base curve.

Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a unit speed arbitrary curve called the base curve in $\mathbb{E}^3$ with Frenet frame $\{T, N, B\}$ and the principal curvatures κ, τ.

If the ratio of curvatures is constant, the curve α is known as *helix* whose tangent line at any point makes a constant angle with a fixed line. On the other hand, when the normal line of curve at any point makes a constant angle with a fixed line, the naming of the curve α changes to *slant helix.* Moreover the characterization of slant helix is briefly obtained as the constant ratio of $\kappa^2\left(\frac{\tau}{\kappa}\right)'$ and $\left(\kappa^2+\tau^2\right)^{3/2}$ [10].

The integral curve called *the principal-direction curve* of base curve α is defined as

$$\gamma(s) = \int N(s)ds$$

[9]. If we compute Frenet frame of the principal-direction curve, then $N, C = \dfrac{N'}{\|N'\|}$ and $W = N \wedge C$ are called the unit principal normal vector, the derivative of principal normal vector and the Darboux vector of the base curve, respectively. Also, for derivative of them the characterization is given,

$$\begin{bmatrix} N'(s) \\ C'(s) \\ W'(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix} \tag{2.1}$$

where the curvature functions $f(s)$ and $g(s)$ can be defined in terms of $\kappa(s), \tau(s)$ which are the principal curvatures of $\alpha(s)$ as followings:

$$f = \sqrt{\kappa^2+\tau^2} \ \text{ and } g = \sigma_\alpha f \tag{2.2}$$

where $\sigma_\alpha(s)$ is the function of $\dfrac{\kappa^2\left(\frac{\tau}{\kappa}\right)'}{\left(\kappa^2+\tau^2\right)^{3/2}}$ [3].

For the future calculations, we give one of the conclusions in [4] as follows:

Corollary 2.1. *If the main curve is slant helix then the related principal direction curve becomes a helix and vice versa.*

Let $\gamma(s)$ be the principal-direction curve of the base curve α and $X : I \longrightarrow \mathbb{E}^3$ be an arbitrary vector along $\gamma(s)$ as $X(s) = x_1N + x_2C + x_3W$ that is $x_1^2 + x_2^2 + x_3^2 = 1$,

$\forall x_i \in \mathbb{R}$, $1 \leq i \leq 3$. Thus, the surface

$$\varphi(s,\upsilon) = \gamma(s) + \upsilon X(s) \tag{2.3}$$

is called a *ruled surface* with the director vector $X(s)$ and the directrix $\gamma(s)$ [1]. If the directrix curve is closed, then the ruled surface is specifically named as closed ruled surface [11]. Then we can give the principal elements for the closed ruled surface $\varphi(s,\upsilon)$ as followings.

Initially, *the pitch of* φ which is satisfied by the X is defined by

$$L_x = -\oint_{(\gamma)} d\upsilon = \oint_{(\gamma)} \langle d\gamma, X\rangle . \tag{2.4}$$

On the other hand, the integral of the real instantaneous vector of the motion will be called *Steiner vector* of the motion and denoted by $\overrightarrow{D} = \oint gds\overrightarrow{N} + \oint fds\overrightarrow{W}$ and *the angle of pitch* of the closed ruled surface is defined by

$$\lambda_X = \left\langle \overrightarrow{D}, X \right\rangle . \tag{2.5}$$

Lastly, *dral* of the ruled surface is defined by,

$$P_X = \frac{\det\left[\dfrac{d\overrightarrow{\gamma}}{ds}, X, \dfrac{dX}{ds}\right]}{\left\|\dfrac{dX}{ds}\right\|^2}. \tag{2.6}$$

In three-dimensional Euclidean space, a smooth surface with zero Gaussian curvature is named as *developable surface*. We know that if a surface is developable, it has to be a ruled surface in $\mathbb{E}^3$, but not vice versa [5]. To illuminate the opposite condition the following theorem can be given.

Theorem 2.1. *Any ruled surface is developable if and if the dral of surface is* $P_X = 0$ [5].

3. Developable Ruled Surface and Slant Helix

In this section, we investigate the developable condition for the ruled surfaces which is generated by the principal-direction curves γ in terms of $\{N, C, W\}$ and give some relations with the base curve α. Moreover, the surfaces are analyzed under the special cases of director vector X.

Theorem 3.1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a unit speed curve and $\gamma(s)$ be the principal-direction curve of α in Euclidean $3-$space $\mathbb{E}^3$. The ruled surface $\varphi(s,\upsilon) = \gamma(s)+\upsilon X(s)$ which is composed of the director $X = x_1N+x_2C+x_3W$ is developable if and only if the principal curvatures of γ satisfies as follows:*

$$\frac{f}{g} = \frac{1-x_1^2}{x_1x_3} \tag{3.1}$$

Proof. The ruled surface $\varphi(s,\upsilon) = \gamma(s) + \upsilon X(s)$ defined by the principal curve $\gamma(s) = \int N(s)ds$ is developable if and only if $\det\left[\dfrac{d\overrightarrow{\gamma}}{ds}, X, \dfrac{dX}{ds}\right] = 0$. Since

$$X = x_1N + x_2C + x_3W,$$

we obtain that

$$\begin{aligned}\frac{dX}{ds} &= x_1N' + x_2C' + x_3W', \\ &= (-x_2f)N + (x_1f - x_3g)C + x_2gW.\end{aligned}$$

Then, the result of the linearity of determinant we have

$$\det[N, x_2C, x_2gW] + \det[N, x_3W, (x_1f - x_3g)C] = 0. \tag{3.2}$$

Thus, we get

$$\frac{f}{g} = \frac{x_2^2 + x_3^2}{x_1x_3} = \frac{1 - x_1^2}{x_1x_3}. \tag{3.3}$$

Conversely, we assume that $f = \dfrac{1 - x_1^2}{x_1x_3}g$ holds. Hence the derivation of directrix vector X is obtained as

$$\frac{dX}{ds} = -x_2N + x_1x_2X. \tag{3.4}$$

and we have

$$\begin{aligned}\det\left[\frac{d\overrightarrow{\gamma}}{ds}, X, \frac{dX}{ds}\right] &= \det[N, X, -x_2N + x_1x_2X] \\ &= 0\end{aligned}$$

This means that the ruled surface $\varphi(s, \upsilon) = \gamma(s) + \upsilon X(s)$ is developable. This completes the proof. □

Corollary 3.1. *Let $\varphi(s, \upsilon) = \gamma(s) + \upsilon X(s)$ be ruled surface. φ is developable surface if and only if the main curve α is a slant helix.*

Proof. From above theorem, if $\varphi(s, \upsilon) = \gamma(s) + \upsilon X(s)$ ruled surface is developable then $\dfrac{f}{g} = \dfrac{1 - x_1^2}{x_1x_3}$ is a constant, namely means that the principal curve $\gamma(s)$ is helix. From the first corollary, the main curve α is a slant helix. On the other hand, when the curve α is a slant helix, then σ_α is a constant. If we combine the equations of the functions $f(s)$ and $g(s)$ in (2.2), we get

$$\frac{f}{g} = \frac{1}{\sigma_\alpha}$$

which is a constant. Hence $\varphi(s, \upsilon)$ occurs as a developable ruled surface. □

In the previous theorem, we see that the developability of ruled surface is closely related with the directrix vector $X(s)$. If the first or the third component of the directrix vanishes, then the theorem can not be answer the situation of ruled surface. In accordance with this purpose we give the following propositions, to illuminate the several deficient respects.

Proposition 3.1. *Let $\varphi(s, \upsilon) = \gamma(s) + \upsilon X(s)$ be ruled surface which is drawn by $X \in Sp\{C, W\}$. φ is developable surface if and only if the main curve α is a helix.*

Proof. If $X = x_1N + x_2C + x_3W \in Sp\{C, W\}$, then $x_1 = 0$ holds. From the Theorem 3.1, it can easily seen that $g(x_2^2 + x_3^2) = 0$. Since $x_2^2 + x_3^2 = 1$, we get $g = 0$. Using equation of $g(s)$ in (2.2) and considering $f(s) \neq 0$, we have $\sigma_\alpha = 0$. From the characterization of slant helix, $\sigma_\alpha = \dfrac{\kappa^2\left(\frac{\tau}{\kappa}\right)'}{\left(\kappa^2 + \tau^2\right)^{3/2}} = 0$ holds. So $\left(\dfrac{\tau}{\kappa}\right)' = 0$

and $\frac{\tau}{\kappa}$ is a constant. Hence, the curve α is a helix. Conversely, we assume that the curve α is a helix. It is obvious that the $\varphi(s,\upsilon)=\gamma(s)+\upsilon X(s)$ ruled surface which is drawn by $X \in Sp\{C,W\}$ is developable. This completes the proof. □

Proposition 3.2. *Let $\varphi(s,\upsilon) = \gamma(s) + \upsilon X(s)$ be ruled surface given by $X \in Sp\{N,C\}$. φ is developable surface if and only if the main curve α is a helix or $\overrightarrow{X}(s) = \overrightarrow{t}$.*

Proof. If $X \in Sp\{N,C\}$, then we have $X = x_1 N + x_2 C$. From the Theorem 3.1, it can easily seen that $gx_2^2 = 0$ which implies $g = 0$ or $x_2 = 0$. Similar to previous proposition, if $g = 0$ we get that the curve α is a helix. If $x_2 = 0$, it is obtained that $x_1 = 1$ and $\overrightarrow{X} = \overrightarrow{t}$. On the other hand, we know from [10] that all helices are slant helices and if $\overrightarrow{X} = \overrightarrow{t}$, then $\frac{f}{g}$ is constant. Thus the ruled surface is developable. □

Now, let us introduce the basic elements of any closed ruled surfaces in terms of the alternative frame $\{N,C,W\}$ which is given for the main curve $\alpha(s)$ by [3, 4] as followings:

Definition 3.1. Let $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a unit speed curve in Euclidean $3-$space $\mathbb{E}^3$. The angles of pitch of the closed ruled surface for the base curve γ and the directrix vectors N, C, W is defined respectively as follows:

$$\begin{aligned} \lambda_N &= \left\langle \overrightarrow{D}, N \right\rangle, \\ \lambda_N &= \oint g ds, \end{aligned} \tag{3.5}$$

where $\overrightarrow{D} = \oint g ds \vec{N} + \oint f ds \vec{W}$. In the same way,

$$\lambda_C = 0$$

and

$$\lambda_W = \oint f ds.$$

Hence the Steiner vector of the motion is

$$\begin{aligned} \overrightarrow{D} &= \lambda_N N + \lambda_C C + \lambda_W W, \\ \overrightarrow{D} &= \lambda_N N + \lambda_W W. \end{aligned} \tag{3.6}$$

Then we calculate the angle of pitch of the closed ruled surface which is consist of the trajectory of X.

$$\lambda_X = \lambda_N x_1 + \lambda_W x_3. \tag{3.7}$$

Definition 3.2. The pitch of the closed ruled surface characterized by base curve γ is defined

$$\begin{aligned} L_N &= \left\langle \oint d\gamma, N \right\rangle, \\ L_N &= \left\langle \oint N ds, N \right\rangle, \\ L_N &= \oint ds. \end{aligned} \tag{3.8}$$

Also,

$$\begin{aligned} L_C &= 0, \\ L_W &= 0. \end{aligned}$$

Moreover, the pitch of the closed ruled surface formed by the trajectory of X is

$$\begin{aligned} L_X &= \oint \langle N ds, x_1 N + x_2 C + x_3 W \rangle, \\ L_X &= x_1 \oint ds, \\ L_X &= x_1 L_N. \end{aligned} \tag{3.9}$$

Corollary 3.2. *Let $\varphi(s,\upsilon) = \gamma(s) + \upsilon X(s)$ be closed ruled surface. φ is developable surface if and only if the curve α is a slant helix with* $\dfrac{f}{g} = \dfrac{\left(L_N^2 - L_X^2\right)\lambda_W}{\left(L_N\lambda_X - L_X\lambda_N\right)L_X}$.

Proof. From Theorem 3.1, if $\varphi(s,\upsilon) = \gamma(s) + \upsilon X(s)$ closed ruled surface is developable, then $\dfrac{1}{\sigma_\alpha} = \dfrac{f}{g} = \dfrac{1 - x_1^2}{x_1 x_3}$ is a constant. From the Eq. (3.7) and (3.9), we can see that $x_1 = \dfrac{L_X}{L_N}$ and $x_3 = \dfrac{\lambda_X - \lambda_N x_1}{\lambda_W}$. If this equality written into place $\dfrac{f}{g} = \dfrac{1 - x_1^2}{x_1 x_3}$, then $\dfrac{f}{g} = \dfrac{1 - \frac{L_X^2}{L_N^2}}{\frac{L_X}{L_N}\frac{\left(L_N\lambda_X - L_X\lambda_N\right)}{L_N\lambda_W}}$. If the necessary arrangements are made, the equation is obtained as $\dfrac{f}{g} = \dfrac{\left(L_N^2 - L_X^2\right)\lambda_W}{\left(L_N\lambda_X - L_X\lambda_N\right)L_X}$.

Conversely, we assume that the curve α is a slant helix with $\dfrac{1}{\sigma_\alpha} = \dfrac{f}{g} = \dfrac{\left(L_N^2 - L_X^2\right)\lambda_W}{\left(L_N\lambda_X - L_X\lambda_N\right)L_X}$. If we use the Eq. (3.7) and (3.9) in last equality, we get $\dfrac{f}{g} = \dfrac{1 - x_1^2}{x_1 x_3}$ is a constant. So from the Corollary 3.1, $\varphi(s,\upsilon) = \gamma(s) + \upsilon X(s)$ closed ruled surface is developable. □

Definition 3.3. Using Eq. (2.6), the dral of the ruled surface can be defined.

The dral of the ruled surface which is drawn by the N line is

$$\begin{aligned} P_N &= \frac{\det\left[\frac{d\vec{\gamma}}{ds}, N, \frac{dN}{ds}\right]}{\left\|\frac{dN}{ds}\right\|^2}, \\ P_N &= 0 \end{aligned} \tag{3.10}$$

Also, the dral of the ruled surface which is drawn by the C line is

$$\begin{aligned} P_C &= \frac{\det\left[N, C, -fN + gW\right]}{\left\|-fN + gW\right\|^2}, \\ P_C &= \frac{\kappa g}{f^2 + g^2}, \end{aligned} \tag{3.11}$$

and the dral of the ruled surface which is drawn by the W line is

$$P_W = \frac{\det[N, W, -gC]}{\|-gC\|^2},$$

$$P_W = \frac{\kappa}{g}. \tag{3.12}$$

Theorem 3.2. *Let $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a unit speed curve in Euclidean $3-$space $\mathbb{E}^3$. There is the following relation between the dral of the ruled surfaces and the curvature of the curve α.*

$$\left(\frac{1}{\sigma_\alpha}\right)^2 = \left(\frac{f}{g}\right)^2 = \frac{P_W}{P_C} - 1 \tag{3.13}$$

Proof. If the rearrange the Eq. (3.11) are made, we have the equation $P_C = \dfrac{\frac{\kappa}{g}}{\left(\frac{f}{g}\right)^2 + 1}$. Then, if the Eq. (3.12) is considered, $P_C = \dfrac{P_W}{\left(\frac{f}{g}\right)^2 + 1}$ holds. So, $\left(\dfrac{f}{g}\right)^2 = \dfrac{P_W}{P_C} - 1$ is obtained. This completes the proof. □

Corollary 3.3. *The curve α is a slant helix if and only if $\dfrac{P_W}{P_C}$ is a constant.*

Proof. From the above theorem, $\left(\dfrac{1}{\sigma_\alpha}\right)^2 = \dfrac{P_W}{P_C} - 1$ is obtained. Hence, if the curve α is a slant helix, then $\dfrac{P_W}{P_C}$ is a constant function.

Conversely, we assume that $\dfrac{P_W}{P_C}$ is a constant. Because of the equation $\left(\dfrac{1}{\sigma_\alpha}\right)^2 = \dfrac{P_W}{P_C} - 1$, $\dfrac{1}{\sigma_\alpha}$ is a constant and the curve α is a slant helix. □

Corollary 3.4. *Let $\varphi(s, \upsilon) = \gamma(s) + \upsilon X(s)$ be closed ruled surface. φ is developable surface if and only if there is the following relation between pitch and angle of pitch of the closed ruled surface and the dral of the closed ruled surfaces.*

$$\frac{P_W}{P_C} = \frac{\left(L_N^2 - L_X^2\right)^2 \lambda_W^2}{\left(L_N \lambda_X - L_X \lambda_N\right)^2 L_X^2} + 1 \tag{3.14}$$

After the theoretical arguments, let us give the following example to complete the theory of developable ruled surfaces.

Example 3.1. Let $\alpha(s) = \left(\dfrac{9}{10}\sin s - \dfrac{1}{90}\sin 9s, -\dfrac{9}{10}\cos s + \dfrac{1}{90}\cos 9s, \dfrac{3}{20}\sin 4s\right)$ be the base curve and the principal-direction curve of α be

$$\gamma(s) = \left(\frac{6}{50}\sin 5s, \frac{6}{50}\cos 5s, \frac{-4}{5}s\right)$$

in Euclidean $3-$space $\mathbb{E}^3$. The ruled surface

$$\varphi(s, \upsilon) = \left(\frac{6}{50}\sin 5s, \frac{-6}{50}\cos 5s + \upsilon\frac{24}{25}\sin 5s, \frac{-4}{5}s - \upsilon\right)$$

which is composed of the directrix curve γ and the director vector $X = \frac{4}{5}N - \frac{3}{5}W$ is developable wherefore the principle curvature of γ and the director vector X correct the condition (3.3) in terms of the alternative moving frame $\{N, C, W\}$.

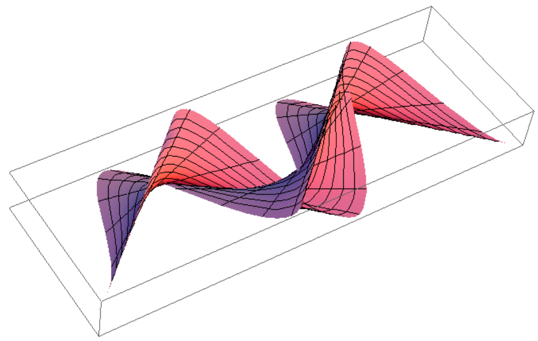

Figure 1. The developable ruled surface $\varphi(s, v)$

4. Conclusion

In this study, the given theory for developable ruled surfaces with helices in terms of the Frenet frame $\{T, N, B\}$ in [5] is obtained with the slant helices approach in terms of the new frame $\{N, C, W\}$ with the help of the integral curve of $N(s)$ which is called the principal-direction curve. On the other hand, if we had used the integral curve of $B(s)$ which is called the binormal-direction curve, then we would have obtained the same result for the frame $\{T, N, B\}$. Let's take our attention to the new orthonormal frame $\{N, C, W\}$ of the main curve α where N and W are the principal normal vector and the Darboux vector of the base curve α, respectively. Moreover the generalization of frame which is obtained from the principle-direction curves is given by Ramis, Uzunoğlu and Yaylı [4] and introduced new curves called N_k-slant helix. So all theorems which is provided that throughout this paper also can be given for the N_k-slant helices.

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