Journal of Mathematical Extension
Vol. 14, No. 1, (2020), 41-59
ISSN: 1735-8299
URL: http://www.ijmex.com

On the System of Difference Equations $x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}(a_n+b_n x_{n-2}y_{n-3})}$, $y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}(\alpha_n+\beta_n y_{n-2}x_{n-3})}$

M. Kara
Aksaray University

Y. Yazlik*
Nevsehir Hacı Bektaş Veli University

Abstract. In this paper, we show that the system of difference equations

$$x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a_n + b_n x_{n-2}y_{n-3}\right)},\ y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}\left(\alpha_n + \beta_n y_{n-2}x_{n-3}\right)},\ n \in \mathbb{N}_0,$$

where the sequences $\forall n \in \mathbb{N}_0$, $(a_n), (b_n), (\alpha_n), (\beta_n)$ and the initial values $x_{-j}, y_{-j}, j \in \{1,2,3\}$ are non-zero real numbers, can be solved in the closed form. For the case when all the sequences $(a_n), (b_n), (\alpha_n), (\beta_n)$ are constant we describe the asymptotic behavior and periodicity of solutions of above system is also investigated.

AMS Subject Classification: 39A10; 39A20; 39A23
Keywords and Phrases: System of difference equation, asymptotic behavior, closed form solution.

1. Introduction

Recently, there has been published quite a lot of studies concerning nonlinear difference equations. One can see this in the sample references [3, 4, 9, 11, 17, 18, 23, 26, 27, 29, 30, 35]. In the meanwhile, the main trend in the theory of difference equations is to expand to two-dimensional

Received: June 2018; Accepted: January 2019
*Corresponding author

or three-dimensional systems of these equations which can be solved in closed-form. See, for example [2, 6, 8,10, 13, 21, 28, 31, 34, 36, 39]. Altough difference equations and difference equations system are very simple in the form, they are highly difficult to characterize the behavior of the solutions of these equations and systems. Therefore, the study of these equations and systems is worth further consideration.

In an earlier paper, Ibrahim et al. in [16] studied the solutions of the rational difference equation

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n\left(a_n + b_n x_{n-1}x_{n-2}\right)},\ n \in \mathbb{N}_0, \tag{1}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are real two-periodic sequences and initial values x_{-2}, x_{-1}, x_0 are nonzero real numbers. Quite recently in [1], Ahmed et al. investigated the periodic character and the form of the solutions of some rational difference equations systems of order-three

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n\left(-1 \pm 1x_{n-1}y_{n-2}\right)},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n\left(\pm 1 \pm y_{n-1}x_{n-2}\right)},\ n \in \mathbb{N}_0, \tag{2}$$

by induction with $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$, and y_0, are nonzero real numbers. When the assumption of $x_n = y_n$ and $x_{-2} = y_{-2}, x_{-1} = y_{-1}, x_0 = y_0$ in system (2), system (2) is reduced special case of Eq. (1).

Our aim in this study is to show that the following difference equations system

$$x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a_n + b_n x_{n-2}y_{n-3}\right)},\ y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}\left(\alpha_n + \beta_n y_{n-2}x_{n-3}\right)},\ n \in \mathbb{N}_0, \tag{3}$$

where the sequences $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (\alpha_n)_{n\in\mathbb{N}_0}, (\beta_n)_{n\in\mathbb{N}_0}$ and the initial values $x_{-j}, y_{-j}, j \in \{1,2,3\}$ are non-zero real numbers, can be solved in closed-form. To do this, we employ appropriate transformation reducing the equation into linear difference equation of order-two. Also, we obtain the forbidden set of the initial values $x_{-j}, y_{-j}, j \in \{1,2,3\}$ for aforementioned system and give a study of the long-term behavior of its solutions when all the sequences $(a_n), (b_n), (\alpha_n), (\beta_n)$, for every $n \in \mathbb{N}_0$, are constant. We emphasize that our study generalizes the results exhibited in [1].

For more works on the topic, see, for example, [5, 14, 15, 19, 22, 24, 25, 32, 33, 37, 38, 40]. Also, see the books [7, 12, 20].

Definition 1.1. (**Periodicity**) *Let $(x_n, y_n)_{n\geqslant -3}$ be solution to difference equation system (3). The solution $(x_n, y_n)_{n\geqslant -3}$ is said to be eventually periodic p if $x_{n+p} = x_n,\ y_{n+p} = y_n$ for all $n \geqslant n_0$. If $n_0 = -3$ is said that the solution is periodic with period p.*

Lemma 1.2. (See [7]) *Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and the sequences y_{2m+i}, $i \in \{0,1\}$, be solutions of the equations*

$$y_{2m+i} = a_{2m+i}y_{2(m-1)+i} + b_{2m+i},\ m \in \mathbb{N}_0. \tag{4}$$

Then, for each fixed $i \in \{0,1\}$ and $m \geqslant -1$, equation (4) has the general solution

$$y_{2m+i} = y_{-2+i}\prod_{j=0}^{m} a_{2j+i} + \sum_{l=0}^{m} b_{2l+i}\prod_{j=l+1}^{m} a_{2j+i}.$$

Further, if $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant and $i \in \{0,1\}$, then

$$y_{2m+i} = \begin{cases} a^{m+1}y_{-2+i} + b\frac{1-a^{m+1}}{1-a}, & \text{if } a \neq 1, \\ y_{-2+i} + b\,(m+1)\,, & \text{if } a = 1. \end{cases}$$

Definition 1.3. (**Forbidden Set**) *Let*

$$\begin{aligned} x_{n+1} &= f\left(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}\right), \\ y_{n+1} &= g\left(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}\right), n \in \mathbb{N}_0, \end{aligned} \tag{5}$$

where $f : \mathbb{R}^6 \to \mathbb{R}$ and $g : \mathbb{R}^6 \to \mathbb{R}$ is given functions, be a system of difference equations and D_f and D_g be the domains of the functions f and g, respectively. The forbidden set of system (5) is given by

$$\begin{aligned} \mathcal{F} \;=\; & \Big\{ (x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0) \in \mathbb{R}^6 : (x_i, y_i) \in D_f \times D_g \\ & \text{for } i = \overline{0, n-1},\ \text{and}\ (x_n, y_n) \notin D_f \times D_g \Big\}. \end{aligned} \tag{6}$$

This set contains all the initial values which causes the undefinable solutions of the system. That is, the initial values chosen from the complement of the forbidden set always produce the well-defined solutions.

2. Closed-Form Solutions of System (3)

Let $\{(x_n, y_n)\}_{n\geqslant -3}$ be a solution of system (3). If at least one of the initial values x_{-i}, y_{-i}, $i = 1, 2, 3$, is equal to zero, then the solution of system (3) is not defined. For example, if $x_{-3} = 0$, then $y_0 = 0$ and so x_1 is not defined. Similarly, if $y_{-3} = 0$, then $x_0 = 0$ and so y_1 is not defined. For $i = 1, 2$, the other cases are similar. On the other hand, if $x_{n_0} = 0$ ($n_0 \in \mathbb{N}_0$), $x_n \neq 0$, for $-3 \leqslant n \leqslant n_0 - 1$, and x_k and y_k are defined for $-3 \leqslant k \leqslant n_0 - 1$, then according to the first equation in (3) we get that $y_{n_0-3} = 0$. If $n_0 - 3 \leqslant -1$, then $y_{-i_0} = 0$, for $i_0 \in \{1, 2, 3\}$. If $n_0 > 2$, then according to the second equation in (3) we have that $y_{n_0-5} = 0$. If $n_0 - 5 \leqslant -1$, then $y_{-i_0} = 0$, for $i_0 \in \{1, 2, 3\}$. Repeating this procedure, we have that $y_{-i_0} = 0$, for $i_0 \in \{1, 2, 3\}$. Similarly, if $y_{n_1} = 0$ ($n_1 \in \mathbb{N}_0$), $y_n \neq 0$, for $-3 \leqslant n \leqslant n_1 - 1$, and x_k and y_k are defined for $-3 \leqslant k \leqslant n_1 - 1$, one can easily show that $x_{-i_1} = 0$, for $i_1 \in \{1, 2, 3\}$. Thus, for every well-defined solutions of system (3), we have that

$$x_n y_n \neq 0, \ n \geqslant -3, \tag{7}$$

if and only if $x_{-i} y_{-i} \neq 0$, for $i \in \{1, 2, 3\}$. Now we give the following theorem describing the form of well-defined solution of system (3).

Theorem 2.1. *Let $\{(x_n, y_n)\}_{n\geqslant -3}$ be a well-defined solution of system (3). Then, for $m \geqslant -1$ and $i \in \{1, 2\}$, we have that*

$$\begin{aligned} x_{4m+2i} &= x_{2i-4} \prod_{j=0}^{m} \left(\frac{y_{-3}}{y_{-1}}\right)^2 \frac{\prod_{s=0}^{2j+i-1} \alpha_{2s+1} + y_{-1}x_{-2} \sum_{l=0}^{2j+i-1} \beta_{2l+1} \prod_{s=l+1}^{2j+i-1} \alpha_{2s+1}}{\prod_{s=0}^{2j+i} a_{2s} + x_{-2}y_{-3} \sum_{l=0}^{2j+i} b_{2l} \prod_{s=l+1}^{2j+i} a_{2s}} \\ &\quad \times \frac{\prod_{s=0}^{2j+i-2} \alpha_{2s+1} + y_{-1}x_{-2} \sum_{l=0}^{2j+i-2} \beta_{2l+1} \prod_{s=l+1}^{2j+i-2} \alpha_{2s+1}}{\prod_{s=0}^{2j+i-1} a_{2s} + x_{-2}y_{-3} \sum_{l=0}^{2j+i-1} b_{2l} \prod_{s=l+1}^{2j+i-1} a_{2s}}, \end{aligned} \tag{8}$$

$$\begin{aligned} x_{4m+2i-1} &= x_{2i-5} \prod_{j=0}^{m} \left(\frac{x_{-1}}{x_{-3}}\right)^2 \frac{\prod_{s=0}^{2j+i-1} \alpha_{2s} + y_{-2}x_{-3} \sum_{l=0}^{2j+i-1} \beta_{2l} \prod_{s=l+1}^{2j+i-1} \alpha_{2s}}{\prod_{s=0}^{2j+i-1} a_{2s+1} + x_{-1}y_{-2} \sum_{l=0}^{2j+i-1} b_{2l+1} \prod_{s=l+1}^{2j+i-1} a_{2s+1}} \\ &\quad \times \frac{\prod_{s=0}^{2j+i-2} \alpha_{2s} + y_{-2}x_{-3} \sum_{l=0}^{2j+i-2} \beta_{2l} \prod_{s=l+1}^{2j+i-2} \alpha_{2s}}{\prod_{s=0}^{2j+i-2} a_{2s+1} + x_{-1}y_{-2} \sum_{l=0}^{2j+i-2} b_{2l+1} \prod_{s=l+1}^{2j+i-2} a_{2s+1}}, \end{aligned} \tag{9}$$

$$y_{4m+2i} = y_{2i-4}\prod_{j=0}^{m}\left(\frac{x_{-3}}{x_{-1}}\right)^2 \frac{\prod_{s=0}^{2j+i-1}a_{2s+1}+x_{-1}y_{-2}\sum_{l=0}^{2j+i-1}b_{2l+1}\prod_{s=l+1}^{2j+i-1}a_{2s+1}}{\prod_{s=0}^{2j+i}\alpha_{2s}+y_{-2}x_{-3}\sum_{l=0}^{2j+i}\beta_{2l}\prod_{s=l+1}^{2j+i}\alpha_{2s}} \times\frac{\prod_{s=0}^{2j+i-2}a_{2s+1}+x_{-1}y_{-2}\sum_{l=0}^{2j+i-2}b_{2l+1}\prod_{s=l+1}^{2j+i-2}a_{2s+1}}{\prod_{s=0}^{2j+i-1}\alpha_{2s}+y_{-2}x_{-3}\sum_{l=0}^{2j+i-1}\beta_{2l}\prod_{s=l+1}^{2j+i-1}\alpha_{2s}}, \quad (10)$$

$$y_{4m+2i-1} = y_{2i-5}\prod_{j=0}^{m}\left(\frac{y_{-1}}{y_{-3}}\right)^2 \frac{\prod_{s=0}^{2j+i-1}a_{2s}+x_{-2}y_{-3}\sum_{l=0}^{2j+i-1}b_{2l}\prod_{s=l+1}^{2j+i-1}a_{2s}}{\prod_{s=0}^{2j+i-1}\alpha_{2s+1}+y_{-1}x_{-2}\sum_{l=0}^{2j+i-1}\beta_{2l+1}\prod_{s=l+1}^{2j+i-1}\alpha_{2s+1}} \times\frac{\prod_{s=0}^{2j+i-2}a_{2s}+x_{-2}y_{-3}\sum_{l=0}^{2j+i-2}b_{2l}\prod_{s=l+1}^{2j+i-2}a_{2s}}{\prod_{s=0}^{2j+i-2}\alpha_{2s+1}+y_{-1}x_{-2}\sum_{l=0}^{2j+i-2}\beta_{2l+1}\prod_{s=l+1}^{2j+i-2}\alpha_{2s+1}}, \quad (11)$$

Proof. By considering (7) and applying the substitution $u_n = \frac{1}{x_n y_{n-1}}$, $v_n = \frac{1}{y_n x_{n-1}}$ for $n \geqslant -2$, then system (3) reduce to the following linear difference equations of order two

$$u_n = a_n u_{n-2} + b_n,\ v_n = \alpha_n v_{n-2} + \beta_n, n \in \mathbb{N}_0, \quad (12)$$

In view of Lemma 1.2, for $i \in \{0,1\}$, the general solutions of equations in (12) are

$$u_{2m+i} = u_{i-2}\prod_{j=0}^{m}a_{2j+i} + \sum_{l=0}^{m}b_{2l+i}\prod_{j=l+1}^{m}a_{2j+i},$$
$$v_{2m+i} = v_{i-2}\prod_{j=0}^{m}\alpha_{2j+i} + \sum_{l=0}^{m}\beta_{2l+i}\prod_{j=l+1}^{m}\alpha_{2j+i},\ m \in \mathbb{N}_0. \quad (13)$$

From the substitution $u_n = \frac{1}{x_n y_{n-1}}$, $v_n = \frac{1}{y_n x_{n-1}}$ for $n \geqslant -2$, we have that

$$x_{2m+i} = \frac{v_{2m+i-1}}{u_{2m+i}}\frac{v_{2m+i-3}}{u_{2m+i-2}}x_{2(m-2)+i},$$
$$y_{2m+i} = \frac{u_{2m+i-1}}{v_{2m+i}}\frac{u_{2m+i-3}}{v_{2m+i-2}}y_{2(m-2)+i},\ m \in \mathbb{N}_0, \quad (14)$$

where $i \in \{0,1\}$, and consequently

$$x_{4m+j} = \frac{v_{4m+j-1}}{u_{4m+j}}\frac{v_{4m+j-3}}{u_{4m+j-2}}x_{4(m-1)+j},\ m \in \mathbb{N}_0$$
$$y_{4m+j} = \frac{u_{4m+j-1}}{v_{4m+j}}\frac{u_{4m+j-3}}{v_{4m+j-2}}y_{4(m-1)+j},\ m \in \mathbb{N}_0, \quad (15)$$

where $j \in \{0,1,2,3\}$, as far as $4m+j \geqslant 1$. From (15), we get that

$$x_{4m+l} = x_{l-4} \prod_{j=0}^{m} \frac{v_{4j+l-1}}{u_{4j+l}} \frac{v_{4j+l-3}}{u_{4j+l-2}}, \ y_{4m+l} = y_{l-4} \prod_{j=0}^{m} \frac{u_{4j+l-1}}{v_{4j+l}} \frac{u_{4j+l-3}}{v_{4j+l-2}}, \tag{16}$$

where $m \geqslant -1$ and $l \in \{1,2,3,4\}$. Employing (13) in (16), we get

$$\begin{aligned} x_{4m+2i} &= x_{2i-4} \prod_{j=0}^{m} \frac{v_{-1} \prod_{s=0}^{2j+i-1} \alpha_{2s+1} + \sum_{l=0}^{2j+i-1} \beta_{2l+1} \prod_{s=l+1}^{2j+i-1} \alpha_{2s+1}}{u_{-2} \prod_{s=0}^{2j+i} a_{2s} + \sum_{l=0}^{2j+i} b_{2l} \prod_{s=l+1}^{2j+i} a_{2s}} \\ &\quad \times \frac{v_{-1} \prod_{s=0}^{2j+i-2} \alpha_{2s+1} + \sum_{l=0}^{2j+i-2} \beta_{2l+1} \prod_{s=l+1}^{2j+i-2} \alpha_{2s+1}}{u_{-2} \prod_{s=0}^{2j+i-1} a_{2s} + \sum_{l=0}^{2j+i-1} b_{2l} \prod_{s=l+1}^{2j+i-1} a_{2s}} \\ &= x_{2i-4} \prod_{j=0}^{m} \frac{(y_{-1}x_{-2})^{-1} \prod_{s=0}^{2j+i-1} \alpha_{2s+1} + \sum_{l=0}^{2j+i-1} \beta_{2l+1} \prod_{s=l+1}^{2j+i-1} \alpha_{2s+1}}{(x_{-2}y_{-3})^{-1} \prod_{s=0}^{2j+i} a_{2s} + \sum_{l=0}^{2j+i} b_{2l} \prod_{s=l+1}^{2j+i} a_{2s}} \\ &\quad \times \frac{(y_{-1}x_{-2})^{-1} \prod_{s=0}^{2j+i-2} \alpha_{2s+1} + \sum_{l=0}^{2j+i-2} \beta_{2l+1} \prod_{s=l+1}^{2j+i-2} \alpha_{2s+1}}{(x_{-2}y_{-3})^{-1} \prod_{s=0}^{2j+i-1} a_{2s} + \sum_{l=0}^{2j+i-1} b_{2l} \prod_{s=l+1}^{2j+i-1} a_{2s}} \\ &= x_{2i-4} \prod_{j=0}^{m} \left(\frac{y_{-3}}{y_{-1}}\right)^2 \frac{\prod_{s=0}^{2j+i-1} \alpha_{2s+1} + y_{-1}x_{-2} \sum_{l=0}^{2j+i-1} \beta_{2l+1} \prod_{s=l+1}^{2j+i-1} \alpha_{2s+1}}{\prod_{s=0}^{2j+i} a_{2s} + x_{-2}y_{-3} \sum_{l=0}^{2j+i} b_{2l} \prod_{s=l+1}^{2j+i} a_{2s}} \\ &\quad \times \frac{\prod_{s=0}^{2j+i-2} \alpha_{2s+1} + y_{-1}x_{-2} \sum_{l=0}^{2j+i-2} \beta_{2l+1} \prod_{s=l+1}^{2j+i-2} \alpha_{2s+1}}{\prod_{s=0}^{2j+i-1} a_{2s} + x_{-2}y_{-3} \sum_{l=0}^{2j+i-1} b_{2l} \prod_{s=l+1}^{2j+i-1} a_{2s}}, \end{aligned}$$

$$\begin{aligned} x_{4m+2i-1} &= x_{2i-5} \prod_{j=0}^{m} \frac{v_{-2} \prod_{s=0}^{2j+i-1} \alpha_{2s} + \sum_{l=0}^{2j+i-1} \beta_{2l} \prod_{s=l+1}^{2j+i-1} \alpha_{2s}}{u_{-1} \prod_{s=0}^{2j+i-1} a_{2s+1} + \sum_{l=0}^{2j+i-1} b_{2l+1} \prod_{s=l+1}^{2j+i-1} a_{2s+1}} \\ &\quad \times \frac{v_{-2} \prod_{s=0}^{2j+i-2} \alpha_{2s} + \sum_{l=0}^{2j+i-2} \beta_{2l} \prod_{s=l+1}^{2j+i-2} \alpha_{2s}}{u_{-1} \prod_{s=0}^{2j+i-2} a_{2s+1} + \sum_{l=0}^{2j+i-2} b_{2l+1} \prod_{s=l+1}^{2j+i-2} a_{2s+1}} \\ &= x_{2i-5} \prod_{j=0}^{m} \frac{(y_{-2}x_{-3})^{-1} \prod_{s=0}^{2j+i-1} \alpha_{2s} + \sum_{l=0}^{2j+i-1} \beta_{2l} \prod_{s=l+1}^{2j+i-1} \alpha_{2s}}{(x_{-1}y_{-2})^{-1} \prod_{s=0}^{2j+i-1} a_{2s+1} + \sum_{l=0}^{2j+i-1} b_{2l+1} \prod_{s=l+1}^{2j+i-1} a_{2s+1}} \\ &\quad \times \frac{(y_{-2}x_{-3})^{-1} \prod_{s=0}^{2j+i-2} \alpha_{2s} + \sum_{l=0}^{2j+i-2} \beta_{2l} \prod_{s=l+1}^{2j+i-2} \alpha_{2s}}{(x_{-1}y_{-2})^{-1} \prod_{s=0}^{2j+i-2} a_{2s+1} + \sum_{l=0}^{2j+i-2} b_{2l+1} \prod_{s=l+1}^{2j+i-2} a_{2s+1}} \\ &= x_{2i-5} \prod_{j=0}^{m} \left(\frac{x_{-1}}{x_{-3}}\right)^2 \frac{\prod_{s=0}^{2j+i-1} \alpha_{2s} + y_{-2}x_{-3} \sum_{l=0}^{2j+i-1} \beta_{2l} \prod_{s=l+1}^{2j+i-1} \alpha_{2s}}{\prod_{s=0}^{2j+i-1} a_{2s+1} + x_{-1}y_{-2} \sum_{l=0}^{2j+i-1} b_{2l+1} \prod_{s=l+1}^{2j+i-1} a_{2s+1}} \\ &\quad \times \frac{\prod_{s=0}^{2j+i-2} \alpha_{2s} + y_{-2}x_{-3} \sum_{l=0}^{2j+i-2} \beta_{2l} \prod_{s=l+1}^{2j+i-2} \alpha_{2s}}{\prod_{s=0}^{2j+i-2} a_{2s+1} + x_{-1}y_{-2} \sum_{l=0}^{2j+i-2} b_{2l+1} \prod_{s=l+1}^{2j+i-2} a_{2s+1}}, \end{aligned}$$

$$
\begin{aligned}
y_{4m+2i} &= y_{2i-4}\prod_{j=0}^{m}\frac{u_{-1}\prod_{s=0}^{2j+i-1}a_{2s+1}+\sum_{l=0}^{2j+i-1}b_{2l+1}\prod_{s=l+1}^{2j+i-1}a_{2s+1}}{v_{-2}\prod_{s=0}^{2j+i}\alpha_{2s}+\sum_{l=0}^{2j+i}\beta_{2l}\prod_{s=l+1}^{2j+i}\alpha_{2s}}\\
&\quad\times\frac{u_{-1}\prod_{s=0}^{2j+i-2}a_{2s+1}+\sum_{l=0}^{2j+i-2}b_{2l+1}\prod_{s=l+1}^{2j+i-2}a_{2s+1}}{v_{-2}\prod_{s=0}^{2j+i-1}\alpha_{2s}+\sum_{l=0}^{2j+i-1}\beta_{2l}\prod_{s=l+1}^{2j+i-1}\alpha_{2s}}\\
&= y_{2i-4}\prod_{j=0}^{m}\frac{(x_{-1}y_{-2})^{-1}\prod_{s=0}^{2j+i-1}a_{2s+1}+\sum_{l=0}^{2j+i-1}b_{2l+1}\prod_{s=l+1}^{2j+i-1}a_{2s+1}}{(y_{-2}x_{-3})^{-1}\prod_{s=0}^{2j+i}\alpha_{2s}+\sum_{l=0}^{2j+i}\beta_{2l}\prod_{s=l+1}^{2j+i}\alpha_{2s}}\\
&\quad\times\frac{(x_{-1}y_{-2})^{-1}\prod_{s=0}^{2j+i-2}a_{2s+1}+\sum_{l=0}^{2j+i-2}b_{2l+1}\prod_{s=l+1}^{2j+i-2}a_{2s+1}}{(y_{-2}x_{-3})^{-1}\prod_{s=0}^{2j+i-1}\alpha_{2s}+\sum_{l=0}^{2j+i-1}\beta_{2l}\prod_{s=l+1}^{2j+i-1}\alpha_{2s}}\\
&= y_{2i-4}\prod_{j=0}^{m}\left(\frac{x_{-3}}{x_{-1}}\right)^{2}\frac{\prod_{s=0}^{2j+i-1}a_{2s+1}+x_{-1}y_{-2}\sum_{l=0}^{2j+i-1}b_{2l+1}\prod_{s=l+1}^{2j+i-1}a_{2s+1}}{\prod_{s=0}^{2j+i}\alpha_{2s}+y_{-2}x_{-3}\sum_{l=0}^{2j+i}\beta_{2l}\prod_{s=l+1}^{2j+i}\alpha_{2s}}\\
&\quad\times\frac{\prod_{s=0}^{2j+i-2}a_{2s+1}+x_{-1}y_{-2}\sum_{l=0}^{2j+i-2}b_{2l+1}\prod_{s=l+1}^{2j+i-2}a_{2s+1}}{\prod_{s=0}^{2j+i-1}\alpha_{2s}+y_{-2}x_{-3}\sum_{l=0}^{2j+i-1}\beta_{2l}\prod_{s=l+1}^{2j+i-1}\alpha_{2s}},
\end{aligned}
$$

$$
\begin{aligned}
y_{4m+2i-1} &= y_{2i-5}\prod_{j=0}^{m}\frac{u_{-2}\prod_{s=0}^{2j+i-1}a_{2s}+\sum_{l=0}^{2j+i-1}b_{2l}\prod_{s=l+1}^{2j+i-1}a_{2s}}{v_{-1}\prod_{s=0}^{2j+i-1}\alpha_{2s+1}+\sum_{l=0}^{2j+i-1}\beta_{2l+1}\prod_{s=l+1}^{2j+i-1}\alpha_{2s+1}}\\
&\quad\times\frac{u_{-2}\prod_{s=0}^{2j+i-2}a_{2s}+\sum_{l=0}^{2j+i-2}b_{2l}\prod_{s=l+1}^{2j+i-2}a_{2s}}{v_{-1}\prod_{s=0}^{2j+i-2}\alpha_{2s+1}+\sum_{l=0}^{2j+i-2}\beta_{2l+1}\prod_{s=l+1}^{2j+i-2}\alpha_{2s+1}}\\
&= y_{2i-5}\prod_{j=0}^{m}\frac{(x_{-2}y_{-3})^{-1}\prod_{s=0}^{2j+i-1}a_{2s}+\sum_{l=0}^{2j+i-1}b_{2l}\prod_{s=l+1}^{2j+i-1}a_{2s}}{(y_{-1}x_{-2})^{-1}\prod_{s=0}^{2j+i-1}\alpha_{2s+1}+\sum_{l=0}^{2j+i-1}\beta_{2l+1}\prod_{s=l+1}^{2j+i-1}\alpha_{2s+1}}\\
&\quad\times\frac{(x_{-2}y_{-3})^{-1}\prod_{s=0}^{2j+i-2}a_{2s}+\sum_{l=0}^{2j+i-2}b_{2l}\prod_{s=l+1}^{2j+i-2}a_{2s}}{(y_{-1}x_{-2})^{-1}\prod_{s=0}^{2j+i-2}\alpha_{2s+1}+\sum_{l=0}^{2j+i-2}\beta_{2l+1}\prod_{s=l+1}^{2j+i-2}\alpha_{2s+1}}\\
&= y_{2i-5}\prod_{j=0}^{m}\left(\frac{y_{-1}}{y_{-3}}\right)^{2}\frac{\prod_{s=0}^{2j+i-1}a_{2s}+x_{-2}y_{-3}\sum_{l=0}^{2j+i-1}b_{2l}\prod_{s=l+1}^{2j+i-1}a_{2s}}{\prod_{s=0}^{2j+i-1}\alpha_{2s+1}+y_{-1}x_{-2}\sum_{l=0}^{2j+i-1}\beta_{2l+1}\prod_{s=l+1}^{2j+i-1}\alpha_{2s+1}}\\
&\quad\times\frac{\prod_{s=0}^{2j+i-2}a_{2s}+x_{-2}y_{-3}\sum_{l=0}^{2j+i-2}b_{2l}\prod_{s=l+1}^{2j+i-2}a_{2s}}{\prod_{s=0}^{2j+i-2}\alpha_{2s+1}+y_{-1}x_{-2}\sum_{l=0}^{2j+i-2}\beta_{2l+1}\prod_{s=l+1}^{2j+i-2}\alpha_{2s+1}}
\end{aligned}
$$

for every $m \geqslant -1$, $i \in \{1, 2\}$. $\square$

The forbidden set of the initial values for system (3) can be given in the following theorem.

Theorem 2.2. *Assume that* $a_n \neq 0,\ b_n \neq 0,\ \alpha_n \neq 0, \beta_n \neq 0,\ n \in \mathbb{N}_0$. *Then the domain of undefinable solutions of system (3) is the set*

$$\mathcal{F} = \bigcup_{m\in\mathbb{N}_0}\bigcup_{i=0}^{1}\Big\{ (x_{-3},x_{-2},x_{-1},y_{-3},y_{-2},y_{-1})\in\mathbb{R}^6:\ x_{i-2}y_{i-3}=\frac{1}{c_m},$$

$$y_{i-2}x_{i-3}=\frac{1}{d_m},\ \textit{where}\ c_m:=-\sum_{j=0}^{m}\frac{b_{2j+i}}{a_{2j+i}}\prod_{l=0}^{j-1}\frac{1}{a_{2l+i}}\neq 0,$$

$$d_m:=-\sum_{j=0}^{m}\frac{\beta_{2j+i}}{\alpha_{2j+i}}\prod_{l=0}^{j-1}\frac{1}{\alpha_{2l+i}}\neq 0\Big\}\bigcup$$

$$\bigcup_{j=1}^{3}\Big\{ (x_{-3},x_{-2},x_{-1},y_{-3},y_{-2},y_{-1})\in\mathbb{R}^6: x_{-j}=0,\ y_{-j}=0\Big\} \quad (17)$$

Proof. At the begining of Section 2, we have acquired that the set

$$\bigcup_{j=1}^{3}\Big\{ (x_{-3},x_{-2},x_{-1},y_{-3},y_{-2},y_{-1})\in\mathbb{R}^6: x_{-j}=0,\ y_{-j}=0\Big\}.$$

belongs to the forbidden set of the initial values for system (3). Now, we assume that $x_n\neq 0$ and $y_n\neq 0$. Note that the system (3) is undefined, when the conditions $a_n+b_nx_{n-2}y_{n-3}=0$ or $\alpha_n+\beta_ny_{n-2}x_{n-3}=0$, that is, $x_{n-2}y_{n-3}=-\frac{a_n}{b_n}$ or $y_{n-2}x_{n-3}=-\frac{\alpha_n}{\beta_n}$, for some $n\in\mathbb{N}_0$, are satisfied (Here we consider that $b_n\neq 0$ and $\beta_n\neq 0$ for every $n\in\mathbb{N}_0$). From this and the substitution $u_n=\frac{1}{x_ny_{n-1}}$, $v_n=\frac{1}{y_nx_{n-1}}$, we get

$$u_{2(m-1)+i}=-\frac{b_{2m+i}}{a_{2m+i}},\ v_{2(m-1)+i}=-\frac{\beta_{2m+i}}{\alpha_{2m+i}} \quad (18)$$

for some $m\in\mathbb{N}_0$ and $i\in\{0,1\}$. Hence, we can determine the forbidden set of the initial values for system (3) by using the substitution $u_n=\frac{1}{x_ny_{n-1}}$, $v_n=\frac{1}{y_nx_{n-1}}$. Now, we consider the functions

$$\begin{aligned} f_{2m+i}(t): &= a_{2m+i}t+b_{2m+i},\\ g_{2m+i}(t): &= \alpha_{2m+i}t+\beta_{2m+i},\ m\in\mathbb{N}_0,\ i\in\{0,1\}, \end{aligned} \quad (19)$$

which correspond to the equations of (12). From (12) and (19), we can write

$$u_{2m+i}=f_{2m+i}\circ f_{2(m-1)+i}\circ\cdots\circ f_i(u_{i-2}), \quad (20)$$

$$v_{2m+i} = g_{2m+i} \circ g_{2(m-1)+i} \circ \cdots \circ g_i \left(v_{i-2}\right), \tag{21}$$

where $m \in \mathbb{N}_0$, and $i \in \{0, 1\}$. By using (18) and implicit forms (20)-(21) and considering $f_{2m+i}^{-1}(0) = -\frac{b_{2m+i}}{a_{2m+i}}$, $g_{2m+i}^{-1}(0) = -\frac{\beta_{2m+i}}{\alpha_{2m+i}}$, for $m \in \mathbb{N}_0$ and $i \in \{0, 1\}$, we have

$$u_{i-2} = f_i^{-1} \circ \cdots \circ f_{2m+i}^{-1}(0), \ v_{i-2} = g_i^{-1} \circ \cdots \circ g_{2m+i}^{-1}(0), \tag{22}$$

where $f_{2m+i}^{-1}(t) = \frac{t-b_{2m+i}}{a_{2m+i}}$, $g_{2m+i}^{-1}(t) = \frac{t-\beta_{2m+i}}{\alpha_{2m+i}}$, $m \in \mathbb{N}_0$, $i \in \{0, 1\}$. From (22), we obtain

$$u_{i-2} = -\sum_{j=0}^{m} \frac{b_{2j+i}}{a_{2j+i}} \prod_{l=0}^{j-1} \frac{1}{a_{2l+i}}, \ v_{i-2} = -\sum_{j=0}^{m} \frac{\beta_{2j+i}}{\alpha_{2j+i}} \prod_{l=0}^{j-1} \frac{1}{\alpha_{2l+i}}$$

for some $m \in \mathbb{N}_0$ and $i \in \{0, 1\}$. This means that if one of the conditions in (22) holds, then m-th iteration or $(m+1)$-th iteration in system (3) can not be calculated. □

3. Case of Constant Coefficients

In this section, we examine the forms of solution and the long-term behavior of the solution of system (3) for the case when a_n, b_n, α_n and β_n are constant, that is $a_n = a, b_n = b, \alpha_n = \alpha$ and $\beta_n = \beta$, for every $n \in \mathbb{N}_0$. Then, the system (3) becomes

$$x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a + bx_{n-2}y_{n-3}\right)}, \ y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}\left(\alpha + \beta y_{n-2}x_{n-3}\right)}, \ n \in \mathbb{N}_0. \tag{23}$$

We start the following theorem describing the form of well-defined solution of system (23).

Theorem 3.1. *Let $\{(x_n, y_n)\}_{n \geqslant -3}$ be a well-defined solution of system (23). Then, for $m \geqslant -1$ and $i \in \{1, 2\}$, we get that*

$$
\begin{aligned}
x_{4m+2i} &= x_{2i-4}\prod_{j=0}^{m}\left(\frac{y_{-3}}{y_{-1}}\right)^2\left(\frac{1-a}{1-\alpha}\right)^2\frac{\alpha^{2j+i}\left((1-\alpha)-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}{a^{2j+i+1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}\\
&\quad\times\frac{\alpha^{2j+i-1}\left((1-\alpha)-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}{a^{2j+i}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b},\\
x_{4m+2i-1} &= x_{2i-5}\prod_{j=0}^{m}\left(\frac{x_{-1}}{x_{-3}}\right)^2\left(\frac{1-a}{1-\alpha}\right)^2\frac{\alpha^{2j+i}\left((1-\alpha)-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}{a^{2j+i}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}\\
&\quad\times\frac{\alpha^{2j+i-1}\left((1-\alpha)-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}{a^{2j+i-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b},\\
y_{4m+2i} &= y_{2i-4}\prod_{j=0}^{m}\left(\frac{x_{-3}}{x_{-1}}\right)^2\left(\frac{1-\alpha}{1-a}\right)^2\frac{a^{2j+i}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{\alpha^{2j+i+1}\left((1-\alpha)-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}\\
&\quad\times\frac{a^{2j+i-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{\alpha^{2j+i}\left((1-\alpha)-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta},\\
y_{4m+2i-1} &= y_{2i-5}\prod_{j=0}^{m}\left(\frac{y_{-1}}{y_{-3}}\right)^2\left(\frac{1-\alpha}{1-a}\right)^2\frac{a^{2j+i}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{\alpha^{2j+i}\left((1-\alpha)-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}\\
&\quad\times\frac{a^{2j+i-1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{\alpha^{2j+i-1}\left((1-\alpha)-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta},
\end{aligned}
$$

when $a \neq 1 \neq \alpha$, *and*

$$
\begin{aligned}
x_{4m+2i} &= x_{2i-4}\prod_{j=0}^{m}\left(\frac{y_{-3}}{y_{-1}}\right)^2\frac{1+y_{-1}x_{-2}\beta(2j+i)}{1+x_{-2}y_{-3}b(2j+i+1)}\,\frac{1+y_{-1}x_{-2}\beta(2j+i-1)}{1+x_{-2}y_{-3}b(2j+i)},\\
x_{4m+2i-1} &= x_{2i-5}\prod_{j=0}^{m}\left(\frac{x_{-1}}{x_{-3}}\right)^2\frac{1+y_{-2}x_{-3}\beta(2j+i)}{1+x_{-1}y_{-2}b(2j+i)}\,\frac{1+y_{-2}x_{-3}\beta(2j+i-1)}{1+x_{-1}y_{-2}b(2j+i-1)},\\
y_{4m+2i} &= y_{2i-4}\prod_{j=0}^{m}\left(\frac{x_{-3}}{x_{-1}}\right)^2\frac{1+x_{-1}y_{-2}b(2j+i)}{1+y_{-2}x_{-3}\beta(2j+i+1)}\,\frac{1+x_{-1}y_{-2}b(2j+i-1)}{1+y_{-2}x_{-3}\beta(2j+i)},\\
y_{4m+2i-1} &= y_{2i-5}\prod_{j=0}^{m}\left(\frac{y_{-1}}{y_{-3}}\right)^2\frac{1+x_{-2}y_{-3}b(2j+i)}{1+y_{-1}x_{-2}\beta(2j+i)}\,\frac{1+x_{-2}y_{-3}b(2j+i-1)}{1+y_{-1}x_{-2}\beta(2j+i-1)},
\end{aligned}
$$

when $a = 1 = \alpha$.

Proof. By using Lemma 1.2 and Theorem 2.1, we can easily obtain the general solution of system (23). □

Now, the long-term behavior of well-defined solution of system (23) can be given in the following theorems.

Theorem 3.2. *Let* $\{(x_n, y_n)\}_{n\geqslant -3}$ *be a well-defined solution of system (23). Assume that* $a \neq -1,\ \alpha \neq -1,\ b\beta \neq 0$. *Then the following results are true.*

(a) *If* $|a|>1, y_{-1}x_{-2}=\frac{1-\alpha}{\beta}=y_{-2}x_{-3},\ x_{-1}y_{-2}\neq\frac{1-a}{b}\neq x_{-2}y_{-3}$, *then* $x_n\to\ 0$ *and* $|y_n|\to\infty$, *as* $n\to\infty$.

(b) *If* $|\alpha|>1, x_{-1}y_{-2}=\frac{1-a}{b}=x_{-2}y_{-3},\ y_{-1}x_{-2}\neq\frac{1-\alpha}{\beta}\neq y_{-2}x_{-3}$, *then* $|x_n|\to\infty$ *and* $y_n\to\ 0$, *as* $n\to\infty$.

(c) *If* $|\frac{a}{\alpha}|>1, x_{-1}y_{-2}\neq\frac{1-a}{b}\neq x_{-2}y_{-3},\ y_{-1}x_{-2}\neq\frac{1-\alpha}{\beta}\neq y_{-2}x_{-3}$, *then* $x_n\to\ 0$ *and* $|y_n|\to\infty$, *as* $n\to\infty$.

(d) *If* $|\frac{a}{\alpha}|<1, x_{-1}y_{-2}\neq\frac{1-a}{b}\neq x_{-2}y_{-3},\ y_{-1}x_{-2}\neq\frac{1-\alpha}{\beta}\neq y_{-2}x_{-3}$, *then* $|x_n|\to\infty$ *and* $y_n\to\ 0$, *as* $n\to\infty$.

(e) *If* $x_{-3}=x_{-1},\ y_{-3}=y_{-1}$; $b=\beta,\ x_{-1}y_{-2}=\frac{1-a}{b}=x_{-2}y_{-3}$, $y_{-1}x_{-2}=\frac{1-\alpha}{\beta}=y_{-2}x_{-3}$, *or* $a=\alpha=0,\ b=\beta$, *then* $x_{4m+j}=x_{j-4}$ *and* $y_{4m+j}=y_{j-4},\ m\in\mathbb{N}_0$, $j=\overline{1,4}$.

(f) *If* $a=\alpha=1,\ x_{-1}=x_{-3},\ y_{-1}=y_{-3},\ b=\beta$, *then the sequences* $x_{4m+2i-1}=x_{2i-5},\ y_{4m+2i-1}=y_{2i-5}$ *and* $(x_{4m+2i})_{m\in\mathbb{N}_0}$, $(y_{4m+2i})_{m\in\mathbb{N}_0}$ *converge for every* $i\in\{1,2\}$.

Proof. Suppose that

$$p_m^{2i}=\left(\frac{y_{-3}}{y_{-1}}\right)^2\left(\frac{1-a}{1-\alpha}\right)^2 \times\frac{\alpha^{2m+i}\left((1-\alpha)-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}{a^{2m+i+1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}\,\frac{\alpha^{2m+i-1}\left((1-\alpha)-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}{a^{2m+i}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b},\tag{24}$$

$$p_m^{2i-1}=\left(\frac{x_{-1}}{x_{-3}}\right)^2\left(\frac{1-a}{1-\alpha}\right)^2 \times\frac{\alpha^{2m+i}\left((1-\alpha)-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}{a^{2m+i}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}\,\frac{\alpha^{2m+i-1}\left((1-\alpha)-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}{a^{2m+i-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b},\tag{25}$$

$$q_m^{2i}=\left(\frac{x_{-3}}{x_{-1}}\right)^2\left(\frac{1-\alpha}{1-a}\right)^2 \times\frac{a^{2m+i}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{\alpha^{2m+i+1}\left((1-\alpha)-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}\,\frac{a^{2m+i-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{\alpha^{2m+i}\left((1-\alpha)-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta},\tag{26}$$

and

$$q_m^{2i-1}=\left(\frac{y_{-1}}{y_{-3}}\right)^2\left(\frac{1-\alpha}{1-a}\right)^2 \times\frac{a^{2m+i}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{\alpha^{2m+i}\left((1-\alpha)-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}\,\frac{a^{2m+i-1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{\alpha^{2m+i-1}\left((1-\alpha)-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta},\tag{27}$$

for $m \geqslant -1$, $i \in \{1,2\}$.
(a) If $|a|>1, y_{-1}x_{-2}=\frac{1-\alpha}{\beta}=y_{-2}x_{-3},\ x_{-1}y_{-2}\neq\frac{1-a}{b}\neq x_{-2}y_{-3}$, from (24), (25), (26) and (27), we have that

$$\lim_{m\to\infty} p_m^{2i} = \lim_{m\to\infty} p_m^{2i-1} = 0, \tag{28}$$

$$\lim_{m\to\infty} |q_m^{2i}| = \lim_{m\to\infty} |q_m^{2i-1}| = \infty. \tag{29}$$

The results follow from (28) and (29) and Theorem 3.1.
(b) If $|\alpha|>1, x_{-1}y_{-2}=\frac{1-a}{b}=x_{-2}y_{-3},\ y_{-1}x_{-2}\neq\frac{1-\alpha}{\beta}\neq y_{-2}x_{-3}$, from (24), (25), (26) and (27), we get that

$$\lim_{m\to\infty} |p_m^{2i}| = \lim_{m\to\infty} |p_m^{2i-1}| = \infty, \tag{30}$$

$$\lim_{m\to\infty} q_m^{2i} = \lim_{m\to\infty} q_m^{2i-1} = 0. \tag{31}$$

The statemant follows easily from (30), (31) and Theorem 3.1.
(c) If $|\frac{a}{\alpha}|>1, x_{-1}y_{-2}\neq\frac{1-a}{b}\neq x_{-2}y_{-3},\ y_{-1}x_{-2}\neq\frac{1-\alpha}{\beta}\neq y_{-2}x_{-3}$, from (24), (25), (26) and (27), we have

$$\lim_{m\to\infty} p_m^{2i} = \lim_{m\to\infty} p_m^{2i-1} = 0, \tag{32}$$

$$\lim_{m\to\infty} |q_m^{2i}| = \lim_{m\to\infty} |q_m^{2i-1}| = \infty. \tag{33}$$

The results can be seen easily from (32), (33) and Theorem 3.1.
(d) If $|\frac{a}{\alpha}|<1, x_{-1}y_{-2}\neq\frac{1-a}{b}\neq x_{-2}y_{-3},\ y_{-1}x_{-2}\neq\frac{1-\alpha}{\beta}\neq y_{-2}x_{-3}$, from (24), (25), (26) and (27), we have

$$\lim_{m\to\infty} |p_m^{2i}| = \lim_{m\to\infty} |p_m^{2i-1}| = \infty, \tag{34}$$

$$\lim_{m\to\infty} q_m^{2i} = \lim_{m\to\infty} q_m^{2i-1} = 0. \tag{35}$$

From (34), (35) and Theorem 3.1, the statement easily follows.
(e) If $x_{-3}=x_{-1},\ y_{-3}=y_{-1};\ \ b=\beta,\ \ x_{-1}y_{-2}=\frac{1-a}{b}=x_{-2}y_{-3}$, $y_{-1}x_{-2}=\frac{1-\alpha}{\beta}=y_{-2}x_{-3}$, or $a=\alpha=0,\ b=\beta$, the result follows from direct calculations and formulas in Theorem 3.1.
(f) Suppose that

$$r_m^{2i} = \left(\frac{y_{-3}}{y_{-1}}\right)^2 \frac{1+y_{-1}x_{-2}\beta(2m+i)}{1+x_{-2}y_{-3}b(2m+i+1)} \frac{1+y_{-1}x_{-2}\beta(2m+i-1)}{1+x_{-2}y_{-3}b(2m+i)}, \tag{36}$$

$$r_m^{2i-1} = \left(\frac{x_{-1}}{x_{-3}}\right)^2 \frac{1 + y_{-2}x_{-3}\beta(2m+i)}{1 + x_{-1}y_{-2}b(2m+i)} \frac{1 + y_{-2}x_{-3}\beta(2m+i-1)}{1 + x_{-1}y_{-2}b(2m+i-1)}, \quad (37)$$

$$s_m^{2i} = \left(\frac{x_{-3}}{x_{-1}}\right)^2 \frac{1 + x_{-1}y_{-2}b(2m+i)}{1 + y_{-2}x_{-3}\beta(2m+i+1)} \frac{1 + x_{-1}y_{-2}b(2m+i-1)}{1 + y_{-2}x_{-3}\beta(2m+i)}, \quad (38)$$

and

$$s_m^{2i-1} = \left(\frac{y_{-1}}{y_{-3}}\right)^2 \frac{1 + x_{-2}y_{-3}b(2m+i)}{1 + y_{-1}x_{-2}\beta(2m+i)} \frac{1 + x_{-2}y_{-3}b(2m+i-1)}{1 + y_{-1}x_{-2}\beta(2m+i-1)}, \quad (39)$$

for $i \in \{1, 2\}$. Employing the Taylor expansion for $(1+x)^{-1}$ in (36) and (38) and from (37) and (39), we get, for each $i \in \{1, 2\}$,

$$\begin{aligned} r_m^{2i} &= \frac{1 + y_{-1}x_{-2}b(2m+i)}{1 + x_{-2}y_{-1}b(2m+i+1)} \frac{1 + y_{-1}x_{-2}b(2m+i-1)}{1 + x_{-2}y_{-1}b(2m+i)} \\ &= \left(1 - \frac{1}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1 - \frac{1}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right)\right) \\ &= 1 - \frac{1}{m} + \mathcal{O}\left(\frac{1}{m^2}\right), \end{aligned} \quad (40)$$

for sufficiently large m,

$$r_m^{2i-1} = \frac{1 + y_{-2}x_{-1}b(2m+i)}{1 + x_{-1}y_{-2}b(2m+i)} \frac{1 + y_{-2}x_{-1}b(2m+i-1)}{1 + x_{-1}y_{-2}b(2m+i-1)} = 1, \quad (41)$$

$$\begin{aligned} s_m^{2i} &= \frac{1 + x_{-1}y_{-2}b(2m+i)}{1 + y_{-2}x_{-1}b(2m+i+1)} \frac{1 + x_{-1}y_{-2}b(2m+i-1)}{1 + y_{-2}x_{-1}b(2m+i)} \\ &= \left(1 - \frac{1}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1 - \frac{1}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right)\right) \\ &= 1 - \frac{1}{m} + \mathcal{O}\left(\frac{1}{m^2}\right) \end{aligned} \quad (42)$$

for sufficiently large m, and

$$s_m^{2i-1} = \frac{1 + x_{-2}y_{-1}b(2m+i)}{1 + y_{-1}x_{-2}b(2m+i)} \frac{1 + x_{-2}y_{-1}b(2m+i-1)}{1 + y_{-1}x_{-2}b(2m+i-1)} = 1. \quad (43)$$

By considering (41), (43) and from (40), (42) and the relations $\sum_{j=1}^{n}(1/j) \to \infty$ as $n \to \infty$, the results easily follow. □

Theorem 3.3. *Let $\{(x_n, y_n)\}_{n\geqslant -3}$ be a well-defined solution of system (23). Assume that $a = \alpha = -1, x_{-1} = x_{-3}, y_{-1} = y_{-3}, b = \beta \neq 0$. Then, $\{(x_n, y_n)\}_{n\geqslant -3}$ is four-periodic.*

Proof. From assumption of this theorem and Theorem 3.1, we have that

$$\begin{aligned} x_{4m+2i} &= x_{2i-4}\prod_{j=0}^{m}\frac{(-1)^{2j+i}\left(2-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}{(-1)^{2j+i+1}\left(2-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}\frac{(-1)^{2j+i-1}\left(2-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}{(-1)^{2j+i}\left(2-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}\\ &= x_{2i-4}\prod_{j=0}^{m}\frac{(-1)^{2j+i}\left(2-y_{-1}x_{-2}b\right)+y_{-1}x_{-2}b}{(-1)^{2j+i+1}\left(2-x_{-2}y_{-1}b\right)+x_{-2}y_{-1}b}\frac{(-1)^{2j+i-1}\left(2-y_{-1}x_{-2}b\right)+y_{-1}x_{-2}b}{(-1)^{2j+i}\left(2-x_{-2}y_{-1}b\right)+x_{-2}y_{-1}b}\\ &= x_{2i-4}, \end{aligned} \tag{44}$$

$$\begin{aligned} x_{4m+2i-1} &= x_{2i-5}\prod_{j=0}^{m}\frac{(-1)^{2j+i}\left(2-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}{(-1)^{2j+i}\left(2-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}\frac{(-1)^{2j+i-1}\left(2-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}{(-1)^{2j+i-1}\left(2-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}\\ &= x_{2i-5}\prod_{j=0}^{m}\frac{(-1)^{2j+i}\left(2-y_{-2}x_{-1}b\right)+y_{-2}x_{-1}b}{(-1)^{2j+i}\left(2-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}\frac{(-1)^{2j+i-1}\left(2-y_{-2}x_{-1}b\right)+y_{-2}x_{-1}b}{(-1)^{2j+i-1}\left(2-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}\\ &= x_{2i-5}, \end{aligned} \tag{45}$$

$$\begin{aligned} y_{4m+2i} &= y_{2i-4}\prod_{j=0}^{m}\frac{(-1)^{2j+i}\left(2-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{(-1)^{2j+i+1}\left(2-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}\frac{(-1)^{2j+i-1}\left(2-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{(-1)^{2j+i}\left(2-y_{-2}x_{-3}\beta\right)+y_{-2}x_{-3}\beta}\\ &= y_{2i-4}\prod_{j=0}^{m}\frac{(-1)^{2j+i}\left(2-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{(-1)^{2j+i+1}\left(2-y_{-2}x_{-1}b\right)+y_{-2}x_{-1}b}\frac{(-1)^{2j+i-1}\left(2-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{(-1)^{2j+i}\left(2-y_{-2}x_{-1}b\right)+y_{-2}x_{-1}b}\\ &= y_{2i-4}, \end{aligned} \tag{46}$$

$$\begin{aligned} y_{4m+2i-1} &= y_{2i-5}\prod_{j=0}^{m}\frac{(-1)^{2j+i}\left(2-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{(-1)^{2j+i}\left(2-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}\frac{(-1)^{2j+i-1}\left(2-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{(-1)^{2j+i-1}\left(2-y_{-1}x_{-2}\beta\right)+y_{-1}x_{-2}\beta}\\ &= y_{2i-5}\prod_{j=0}^{m}\frac{(-1)^{2j+i}\left(2-x_{-2}y_{-1}b\right)+x_{-2}y_{-1}b}{(-1)^{2j+i}\left(2-y_{-1}x_{-2}b\right)+y_{-1}x_{-2}b}\frac{(-1)^{2j+i-1}\left(2-x_{-2}y_{-1}b\right)+x_{-2}y_{-1}b}{(-1)^{2j+i-1}\left(2-y_{-1}x_{-2}b\right)+y_{-1}x_{-2}b}\\ &= y_{2i-5}, \end{aligned} \tag{47}$$

for $m \geqslant -1$ and $i \in \{1, 2\}$. From (44)-(47), the result easily follows. □

Theorem 3.4. *Let $\{(x_n, y_n)\}_{n\geqslant -3}$ be a well-defined solution of system (23). Assume that $a\alpha \neq 0, a = \alpha,\ b = \beta = 0,\ x_{-1} = x_{-3}, y_{-1} = y_{-3}$. Then, the following results are true.*

(a) *If $|a| > 1$, then $x_{4m+2i} \to 0$ and $y_{4m+2i} \to 0$, for $i \in \{1, 2\}$, as $m \to \infty$.*

(b) *If* $|a| < 1$, $|x_{4m+2i}| \to \infty$ *and* $|y_{4m+2i}| \to \infty$, *for* $i \in \{1,2\}$, *as* $m \to \infty$.

(c) *If* $a = 1$, $\{(x_n, y_n)\}_{n \geqslant -3}$ *is four-periodic,*

(d) *If* $a = -1$, $\{(x_n, y_n)\}_{n \geqslant -3}$ *is four-periodic.*

Proof. From assumption of this theorem and Theorem 3.1, we have that

$$\begin{aligned} x_{4m+2i} &= x_{2i-4} \prod_{j=0}^{m} \frac{(1-a)a^{2j+i}}{(1-a)a^{2j+i+1}} \frac{(1-a)a^{2j+i-1}}{(1-a)a^{2j+i}} \\ &= x_{2i-4} \prod_{j=0}^{m} \frac{1}{a^2}, \end{aligned} \tag{48}$$

$$\begin{aligned} x_{4m+2i-1} &= x_{2i-5} \prod_{j=0}^{m} \frac{(1-a)a^{2j+i}}{(1-a)a^{2j+i}} \frac{(1-a)a^{2j+i-1}}{(1-a)a^{2j+i-1}} \\ &= x_{2i-5}, \end{aligned} \tag{49}$$

$$\begin{aligned} y_{4m+2i} &= y_{2i-4} \prod_{j=0}^{m} \frac{(1-a)a^{2j+i}}{(1-a)a^{2j+i+1}} \frac{(1-a)a^{2j+i-1}}{(1-a)a^{2j+i}} \\ &= y_{2i-4} \prod_{j=0}^{m} \frac{1}{a^2}, \end{aligned} \tag{50}$$

$$\begin{aligned} y_{4m+2i-1} &= y_{2i-5} \prod_{j=0}^{m} \frac{(1-a)a^{2j+i}}{(1-a)a^{2j+i}} \frac{(1-a)a^{2j+i-1}}{(1-a)a^{2j+i-1}} \\ &= y_{2i-5}. \end{aligned} \tag{51}$$

From (48)-(51) the results in (a), (b) and (d) can be seen easily. From Theorem 3.1 and the assumption of this theorem, the statement (c) easily follow. □

References

[1] A. M. Ahmed and E. M. Elsayed, The expressions of solutions and the periodicity of some rational difference equations systems, *Journal of Applied Mathematics and Informatics*, 34 (1-2) (2016), 35-48.

[2] E. O. Alzahrani, M. M. El-Dessoky, E. M. Elsayed, and Y. Kuang, Solutions and properties of some degernerate systems of difference equations, *Journal of Computational Analysis and Applications*, 18 (2015), 321-333.

[3] A. Andruch and S. M. Migda, Further properties of the rational recursive sequence $x_{n+1} = \frac{ax_{n-1}}{b+cx_n x_{n-1}}$, *Opuscula Math.*, 26 (2006), 387-394.

[4] C. Cinar, On the positive solutions of difference equation $x_{n+1} = \frac{x_{n-1}}{1+x_n x_{n-1}}$, *Applied Mathematics and Computation*, 150 (1) (2004), 21-24.

[5] I. Dekkar, N. Touafek, and Y. Yazlik, Global stability of a third-order nonlinear system of difference equations with period-two coefficients, *Racsam. Rev. R. Acad. A.*, 111 (2017), 325-347.

[6] Q. Din, On a system of fourth-order rational difference equations, *Acta Univ. Apulensis, Math. Inform.*, 39 (2014), 137-150.

[7] S. Elaydi, *An Introduction to Difference Equations*, Springer, New York (1996).

[8] M. M. El-Dessoky, E. M. Elsayed, and M. Alghamdi, Solutions and periodicity for some systems of fourth order rational difference equations, *Journal Computational Analysis and Applications*, 18 (1) (2015), 179-194.

[9] H. El-Metwally and E. M. Elsayed, Qualitative study of solutions of some difference equatiaons, *Abstract and Applied Analysis*, Article ID 248291, (2012), 16 pages.

[10] E. M. Elsayed, Solution for systems of difference equations of rational form of order two, *Computational and Applied Mathematics*, 33 (2014), 751-765.

[11] E. M. Elsayed, Expression and behavior of the solutions of some rational recursive sequences, *Mathematical Methods in the Applied Sciences*, 39 (2016), 5682-5694.

[12] E. A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations Advances in Discrete Mathematics and Applications*, vol. 4, Chapman & Hall/CRC, London (2005).

[13] N. Haddad, N. Touafek, and J. F. T. Rabago, Solution form of a higher-order system of difference equations and dynamical behavior of its special case, *Mathematical Methods in the Applied Science*, 40 (10) (2017), 3599-3607.

[14] N. Haddad, N. Touafek, and J. F. T. Rabago, Well-defined solutions of a system of difference equations, *Journal of Applied Matehamtics and Computing*, 56 (1-2) (2018), 439-458.

[15] Y. Halim, N. Touafek, and Y. Yazlik, Dynamic behavior of a second-order nonlinear rational difference equation, *Turkish Journal of Mathematics*, 39 (6) (2015), 1004-1018.

[16] T. F. Ibrahim and N. Touafek, On a third order rational difference equation with variable coefficients, *Dynamics of Discrete and Impulsive Systems Series B Applications Algorithms*, 20 (2013), 251-264.

[17] T. F. Ibrahim, On the third order rational difference equation $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a+bx_n x_{n-2})}$, *International Journal of Contemporary Mathematical Sciences*, 4 (2009), 1321-1334.

[18] A. Khaliq and E. M. Elsayed, The dynamics and solution of some difference equations, *Journal of Nonlinear Science and Applications*, 9 (2016), 1052-1063.

[19] A. Q. Khan, Q. Din, M. N. Qureshi, and T. F. Ibrahim, Global behavior of an anti-competitive system of fourth-order rational difference equations, *Computational Ecology and Software*, 4 (1) (2014), 35-46.

[20] M. R. S. Kulenovic, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman & Hall/CRC Press, London/Boca Raton (2002).

[21] A. S. Kurbanli, C. Cinar, and I. Yalcinkaya, Instructions for authors, *On the behavior of positive solutions of the system of rational difference equations* $x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1}+1}, y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1}+1}$, 53 (2011), 1261-1267.

[22] I. Okumus and Y. Soykan, Dynamical behavior of a system of three-dimensional nonlinear difference equations, *Advances in Difference Equations*, 2018 (223) (2018), 1-15.

[23] Ö. Öcalan, Oscillation of nonlinear difference equations with several coefficients, *Commun. Math. Anal.*, 4 (1) (2008), 35-44.

[24] A. Y. Özban, On the positive solutions of the system of rational difference equations $x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}$, *Journal of Mathematical Analysis and Applications*, 323 (2006), 26-32.

[25] G. Papaschinopoulos and G. Stefanidou, Asymptotic behavior of the solutions of a class of rational difference equations, *International Journal of Difference Equations*, 5 (2) (2010), 233-249.

[26] A. Raouf, Global behavior of the rational riccati difference equation of order two: the general case, *Journal of Difference Equations and Applications*, 18 (2012), 947-961.

[27] S. Stević, J. Diblík, B. Iričanin, and Z. Šmarda, Solvability of nonlinear difference equations of fourth order, *Electronic Journal of Differential Equations*, 2014 (64) (2014), 1-14.

[28] S. Stević, B. Iričanin, and Z. Šmarda, On a close to symmetric system of difference equations of second order, *Advances in Difference Equations*, 264 (2015), 1-17.

[29] D. T. Tollu, Y. Yazlik, and N. Taşkara, On the solutions of two special types of Riccati difference equation via Fibonacci numbers, *Advances in Difference Equations*, 174 (2013).

[30] D. T. Tollu, Y. Yazlik, and N. Taskara, On a solvable nonlinear difference equation of higher order, *Turkish Journal of Mathematics*, 42 (4) (2018), 1765-1778.

[31] D. T. Tollu, Y. Yazlik, and N. Taşkara, On fourteen solvable systems of difference equations, *Applied Mathematics and Computaion*, 233 (2014), 310-319.

[32] N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, *Bull. Math. Soc. Sci. Math. Roumanie Tome*, 55 (103) (2012), 217-224.

[33] I. Yalcinkaya, On the Difference Equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$, *Discrete Dynamics in Nature and Society*, Article ID 805460, (2008), 8 pages.

[34] X. Yang, On the system of rational difference equations $x_n = A + \frac{y_{n-1}}{x_{n-p}y_{n-q}}, y_n = A + \frac{x_{n-1}}{x_{n-r}y_{n-s}}$, *Journal of Mathematical Analysis and Applications*, 307 (2006), 305-311.

[35] Y. Yazlik, On the solutions and behavior of rational difference equations, *Journal of Computational Analysis and Applications*, 17 (3) (2014), 584-594.

[36] Y. Yazlik, E. M. Elsayed, and N. Taşkara, On the behaviour of the solutions of difference equation systems, *Journal of Computational Analysis and Applications*, 16 (5) (2014), 932-941.

[37] Y. Yazlik, D. T. Tollu, and N. Taşkara, On the behaviour of solutions for some systems of difference equations, *Journal of Computational Analysis and Applications*, 18 (1) (2015), 166-178.

[38] Y. Yazlik, D. T. Tollu, and N. Taşkara, On the solutions of a max-type difference equation system, *Mathematical Methods in the Applied Sciences*, 38 (17) (2015), 4388-4410.

[39] Y. Yazlik, D. T. Tollu, and N. Taşkara, On the solutions of a three-dimensional system of difference equations, *Kuwait Journal of Sciences*, 43 (1) (2016), 95-111.

[40] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = \frac{Ax_n+(\beta x_n+\gamma x_{n-k})}{Bx_n+Cx_{n-k}}$, *Communications on Applied Nonlinear Analysis*, 16 (2009), 91-106.

Merva Kara
Lecturer of Mathematics
Ortaköy Vocational School
Aksaray University
Aksaray, Turkey
E-mail: mervekara@aksaray.edu.tr

Yasin Yazlik
Associate Professor of Mathematics
Department of Mathematics
Nevsehir Hacı Bektaş Veli University
Nevsehir, Turkey
E-mail: yyazlik@nevsehir.edu.tr