

# Septic B-Spline Collocation Method for the Numerical Solution of the Modified Equal Width Wave Equation

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## Abstract

Numerical solutions of the modified equal width wave equation are obtained by using collocation method with septic B-spline finite elements with three different linearization techniques. The motion of a single solitary wave, interaction of two solitary waves and birth of solitons are studied using the proposed method. Accuracy of the method is discussed by computing the numerical conserved laws error norms  $L_2$  and  $L_\infty$ . The numerical results show that the present method is a remarkably successful numerical technique for solving the MEW equation. A linear stability analysis shows that this numerical scheme, based on a Crank Nicolson approximation in time, is unconditionally stable.

**Keywords:** Finite Element, Septic B-Spline, Collocation, Soliton, Solitary Waves

## 1. Introduction

This study is concerned with the numerical solution, using septic B-spline functions in collocation method, of the modified equal width wave (MEW) equation, which was introduced by Morrison *et al.* [1] as a model equation to describe the nonlinear dispersive waves. Few analytical solutions of the MEW equation are known. Thus numerical solutions of the MEW equation can be important and comparison between analytic solution can be made. Many methods have been proposed to solve the EW and MEW equation. Gardner and Gardner solved the EW equation with the Galerkin's method using cubic B-splines as a trial and test function [2,3] and a Petrov-Galerkin method using quadratic B-spline element [4]. Zaki considered the solitary wave interactions for the MEW equation by Petrov-Galerkin method using quintic B-spline finite elements [5] and obtained the numerical solution of the EW equation by using least-squares method [6]. Wazwaz investigated the MEW equation and two of its variants by the tanh and the sine-cosine methods [7]. Esen applied a lumped Galerkin method based on quadratic B-spline finite element has been used for solving the EW and MEW equation [8,9]. Saka proposed algorithms for the numerical solution of the MEW equation using quintic B-spline collocation method [10]. A solution based on a collocation method incorporated cubic B-splines is investigated by Dağ and Saka [11].

Variational iteration method is introduced to solve the MEW equation by Junfeng Lu [12]. Hamdi *et al.* [13] derived exact solitary wave solutions of the generalized EW equation using Maple software. D. J. Evans and K. R. Raslan [14] studied the generalized EW equation by using collocation method based on quadratic B-splines to obtain the numerical solutions of a single solitary waves, and the birth of solitons.

The modified equal width wave equation which is as a model for non-linear dispersive waves, considered here has the normalized form [1]

$$U_t + 3U^2U_x - \mu U_{xxt} = 0 \quad (1)$$

with the physical boundary conditions  $U \rightarrow 0$  as  $x \rightarrow \pm\infty$ , where  $t$  is time and  $x$  is the space coordinate,  $\mu$  is a positive parameter. For this study boundary conditions are chosen

$$\begin{aligned} U(a,t) &= 0, U(b,t) = 0, \\ U_x(a,t) &= 0, U_x(b,t) = 0, \\ U_{xxx}(a,t) &= 0, U_{xxx}(b,t) = 0, \end{aligned} \quad (2)$$

and the initial condition as

$$U(x,0) = f(x), \quad a \leq x \leq b$$

where  $f(x)$  is a localized disturbance inside the considered interval.

### 2. Septic B-Spline Collocation Method

The interval  $[a, b]$  is partitioned into uniformly sized finite elements by the knots  $x_i$  such that

$a = x_0 < x_1 \dots < x_N = b$  and  $h = (x_{i+1} - x_i)$ . The septic B-splines  $\phi_i(x)$ ,  $(i = -3(1) N + 3)$ , at the knots  $x_i$  are defined over the interval  $[a, b]$  as [15],

$$\phi_i(x) = \frac{1}{h^7} \begin{cases} (x - x_{i-4})^7, & x \in [x_{i-4}, x_{i-3}], \\ (x - x_{i-4})^7 - 8(x - x_{i-3})^7, & x \in [x_{i-3}, x_{i-2}], \\ (x - x_{i-4})^7 - 8(x - x_{i-3})^7 + 28(x - x_{i-2})^7, & x \in [x_{i-2}, x_{i-1}], \\ (x - x_{i-4})^7 - 8(x - x_{i-3})^7 + 28(x - x_{i-2})^7 - 56(x - x_{i-1})^7, & x \in [x_{i-1}, x_i], \\ (x_{i+4} - x)^7 - 8(x_{i+3} - x)^7 + 28(x_{i+2} - x)^7 - 56(x_{i+1} - x)^7, & x \in [x_i, x_{i+1}], \\ (x_{i+4} - x)^7 - 8(x_{i+3} - x)^7 + 28(x_{i+2} - x)^7, & x \in [x_{i+1}, x_{i+2}], \\ (x_{i+4} - x)^7 - 8(x_{i+3} - x)^7, & x \in [x_{i+2}, x_{i+3}], \\ (x_{i+4} - x)^7, & x \in [x_{i+3}, x_{i+4}], \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The set of splines,  $\{\phi_{-3}(x), \phi_{-2}(x), \dots, \phi_{N+3}(x)\}$  forms a basis for functions defined over  $[a, b]$ . The numerical solution  $U_N(x, t)$  to  $U(x, t)$  takes the form

$$U_N(x, t) = \sum_{i=-3}^{N+3} \delta_i(t) \phi_i(x) \tag{4}$$

where  $\delta_i$  are unknown, time dependent quantities to be

determined from the boundary and collocation conditions and  $\phi_i(x)$  are septic B-spline. Each septic B-spline covers 8 elements thus each element  $[x_i, x_{i+1}]$  is covered by 8 splines. A typical finite interval  $[x_i, x_{i+1}]$  is mapped to the interval  $[0, 1]$  by a local coordinate transformation defined by  $h\xi = x - x_i$ ,  $0 \leq \xi \leq 1$ . Therefore septic B-splines (3) in terms of  $\xi$  over  $[0, 1]$  can be given as

$$\begin{aligned} \phi_{-3} &= 1 - 7\xi + 21\xi^2 - 35\xi^3 + 35\xi^4 - 21\xi^5 + 7\xi^6 - \xi^7 \\ \phi_{-2} &= 120 - 392\xi + 504\xi^2 - 280\xi^3 + 84\xi^5 - 42\xi^6 + 7\xi^7 \\ \phi_{-1} &= 1191 - 1715\xi + 315\xi^2 + 665\xi^3 - 315\xi^4 - 105\xi^5 + 105\xi^6 - 21\xi^7 \\ \phi_i &= 2416 - 1680\xi + 560\xi^4 - 140\xi^6 + 35\xi^7 \\ \phi_{i+1} &= 1191 + 1715\xi + 315\xi^2 - 665\xi^3 - 315\xi^4 + 105\xi^5 + 105\xi^6 - 35\xi^7 \\ \phi_{i+2} &= 120 + 392\xi + 504\xi^2 + 280\xi^3 - 84\xi^5 - 42\xi^6 + 21\xi^7 \\ \phi_{i+3} &= 1 + 7\xi + 21\xi^2 + 35\xi^3 + 35\xi^4 + 21\xi^5 + 7\xi^6 - 7\xi^7 \\ \phi_{i+4} &= \xi^7. \end{aligned} \tag{5}$$

Since all splines apart from  $\phi_{i-3}(x), \phi_{i-2}(x), \dots, \phi_{i+3}(x)$  are zero over the element  $[0, 1]$ . For the problem the finite elements are identified

with the interval  $[x_i, x_{i+1}]$ . Using the nodal values  $U_i, U'_i, U''_i$  and  $U'''_i$  are given in terms of the parameter  $\delta_i$  by:

$$\begin{aligned} U_i &= \delta_{i+3} + 120\delta_{i+2} + 1191\delta_{i+1} + 2416\delta_i + 1191\delta_{i-1} + 120\delta_{i-2} + \delta_{i-3}, \\ hU'_i &= 7(\delta_{i+3} + 56\delta_{i+2} + 245\delta_{i+1} - 245\delta_{i-1} - 56\delta_{i-2} - \delta_{i-3}), \\ h^2U''_i &= 42(\delta_{i+3} + 24\delta_{i+2} + 15\delta_{i+1} - 80\delta_i + 15\delta_{i-1} + 24\delta_{i-2} + \delta_{i-3}), \\ h^3U'''_i &= 210(\delta_{i+3} + 8\delta_{i+2} - 19\delta_{i+1} + 19\delta_{i-1} - 8\delta_{i-2} - \delta_{i-3}), \end{aligned} \tag{6}$$

and the variation of  $U$  over the element  $[x_i, x_{i+1}]$  is given by

$$U = \sum_{i=-3}^{N+3} \delta_i \phi_i$$

We now identify the collocation points with the knots and use (6) to evaluate  $U_i$  and its space derivatives in (1). This leads to a set of ordinary differential equations of the form

$$\begin{aligned} & \dot{\delta}_{i+3} + 120\dot{\delta}_{i+2} + 1191\dot{\delta}_{i+1} + 2416\dot{\delta}_i + 1191\dot{\delta}_{i-1} + 120\dot{\delta}_{i-2} + \dot{\delta}_{i-3} - 21\frac{1}{h}Z_i(\delta_{i+3} + 56\delta_{i+2} + 245\delta_{i+1} - 245\delta_{i-1} - 56\delta_{i-2} - \delta_{i-3}) \\ & - 42\frac{\mu}{h^2}(\delta_{i+3} + 24\delta_{i+2} + 15\delta_{i+1} - 80\delta_i + 15\delta_{i-1} + 24\delta_{i-2} + \delta_{i-3}) = 0, \end{aligned} \tag{7}$$

where

$$Z_i = (\delta_{i+3} + 120\delta_{i+2} + 1191\delta_{i+1} + 2416\delta_i + 1191\delta_{i-1} + 120\delta_{i-2} + \delta_{i-3})^2.$$

If time parameters  $\delta_i$  and its time derivatives  $\dot{\delta}_i$  in (7) are discretized by the Crank-Nicolson formula and usual finite difference approximation, respectively:

$$\delta_i = \frac{1}{2}(\delta^n + \delta^{n+1}), \quad \dot{\delta}_i = \frac{\delta^{n+1} - \delta^n}{\Delta t}$$

We obtain a recurrence relationship between two time levels  $n$  and  $n + 1$  relating two unknown parameters  $\delta_i^{n+1}, \delta_i^n$

$$\begin{aligned} & \gamma_{i1}\delta_{i+3}^{n+1} + \gamma_{i2}\delta_{i+2}^{n+1} + \gamma_{i3}\delta_{i+1}^{n+1} + \gamma_{i4}\delta_i^{n+1} + \gamma_{i5}\delta_{i-1}^{n+1} + \gamma_{i6}\delta_{i-2}^{n+1} + \gamma_{i7}\delta_{i-3}^{n+1} \\ & = \gamma_{i1}\delta_{i+3}^n + \gamma_{i2}\delta_{i+2}^n + \gamma_{i3}\delta_{i+1}^n + \gamma_{i4}\delta_i^n + \gamma_{i5}\delta_{i-1}^n + \gamma_{i6}\delta_{i-2}^n + \gamma_{i7}\delta_{i-3}^n \end{aligned} \tag{8}$$

where

$$\begin{aligned} \gamma_{i1} &= (1 + EZ_i - M), \quad \gamma_{i2} = (120 + 56EZ_i - 24M), \quad \gamma_{i3} = (1191 + 245EZ_i - 15M), \\ \gamma_{i4} &= (2416 + 80M), \quad \gamma_{i5} = (1191 - 245EZ_i - 15M), \quad \gamma_{i6} = (120 - 56EZ_i - 24M), \\ \gamma_{i7} &= (1 - EZ_i - M) \quad i = 0, 1, \dots, N, \\ E &= \frac{21}{2h}\Delta t, \quad M = \frac{42}{h^2}\mu. \end{aligned}$$

For the first linearization, we suppose that the quantity  $U$  in the non-linear term  $U^2U_x$  to be locally constant. This is equivalent to assuming that in (7) all  $Z_i$  are equal to a local constant. Furthermore, we can write the nonlinear term

$$U^2U_x = UUU_x \tag{9}$$

and apply the Rubin and Graves [16] linearization technique

$$(UU_x)^{n+1} = U^{n+1}U_x + U^nU_x^{n+1} - U^nU_x^n \tag{10}$$

to the  $UU_x$  term in (8) and we can also apply the Caldwell and Smith [17] linearization technique

$$(UU_x)^{n+1} = \frac{1}{2}(U^{n+1}U_x + U^nU_x^{n+1}) \tag{11}$$

to the  $UU_x$  term in (8). The system (8) consists of  $N + 1$  linear equation in  $N + 7$  unknowns  $(\delta_{-3}, \delta_{-2}, \dots, \delta_{N+2}, \delta_{N+3})^T$ . To obtain a unique solution to this system we need 6 additional constraints. These are obtained from the boundary conditions and can be used to eliminate  $\delta_{-3}, \delta_{-2}, \delta_{-1}$  and  $\delta_{N+1}, \delta_{N+2}, \delta_{N+3}$  from the set (2) which then becomes a matrix equation for the  $N + 1$  unknowns  $\delta = (\delta_0, \delta_1, \dots, \delta_N)^T$  of the form  $A\delta^{n+1} = B\delta^n$ . The matrices  $A$  and  $B$  are septa-diagonal  $(N + 1) \times (N + 1)$  matrices and so are easily solved by septa-diagonal algorithm.

### 3. Initial State

To start evolution of the vector of parameters  $\delta^n, \delta^0$  can be determined from the boundary conditions and the initial condition  $U(x, 0)$ . So we can rewrite approximation (4) for the initial condition

$$U_N(x, 0) = \sum_{i=-3}^{N+3} \delta_i(0)\phi_i(x),$$

where parameters  $\delta_i^0$  will be determined. To determine the parameters  $\delta^0 = (\delta_{-3}^0, \delta_{-2}^0, \dots, \delta_{N+2}^0, \delta_{N+3}^0)$ , we require the initial numerical approximation  $U_N(x, 0)$  to satisfy the following conditions:

- 1) it must agree with the exact initial condition  $U(x, 0)$  at the knots  $x_i$ .
- 2) the first, second and third derivatives of the approximate initial condition agree with those of the exact initial conditions at both ends of the range. These two conditions can be expressed as:

$$\begin{aligned} U_N(x, 0) &= U_N(x_i, 0), \quad 0 \leq i \leq N, \\ U_{Nx}(a, 0) &= U_{Nx}(b, 0) = 0, \\ U_{Nxx}(a, 0) &= U_{Nxx}(b, 0) = 0, \\ U_{Nxxx}(a, 0) &= U_{Nxxx}(b, 0) = 0. \end{aligned} \tag{12}$$

The above conditions lead to  $K\delta^0 = b$  matrix equation, which is solved by using a variant of Thomas algorithm.



val  $0 \leq x \leq 80$ . The computations are done until time  $t = 20$  and in this length we find error norms  $L_2, L_\infty$  and numerical invariants  $C_1, C_2, C_3$  at various times. In **Table 1** we compare the values of the invariants and error norms obtained using the present method with the three different approximation and the results of [5,14, 18,19] at different times. We can easily see from the this table that the error norms  $L_2$  and  $L_\infty$  are obtained sufficiently small and the quantities in the variants remain almost constant during the computer run for the first and second linearization techniques but we can not say the same for the third linearization technique. For the first and second linearization, the numerical values of invariants are  $C_1 = 0.7853966, C_2 = 0.1667641, C_3 = 0.0052083$  and for the third linearization numerical values of invariants are  $C_1 = 0.7855405, C_2 = 0.1667641, C_3 = 0.0052144$  at the  $t = 20$ . **Figure 1** shows that the proposed method perform the motion of propagation of a solitary wave satisfactorily, which moved to the right at a constant speed and preserved its amplitude and shape with in-

creasing time as expected. Amplitude is 0.249999 at  $t = 0$  which is located at  $x = 30$ , while it is 0.249922 at  $t = 20$  which is located at  $x = 30.6$ . The absolute difference in amplitudes at times  $t = 0$  and  $t = 20$  is  $7.7 \times 10^{-5}$  so that there is a little change between amplitudes.

### 7. Interaction of Two Solitary Waves

For this problem, we consider (1) with boundary conditions  $U \rightarrow 0$  as  $x \rightarrow \pm\infty$ , interaction of two positive solitary waves is studied by using the initial condition

$$U(x, 0) = \sum_{j=1}^2 A_j \operatorname{sech} \left( k [x - x_j] \right).$$

where  $k = \sqrt{1/\mu}$ .

We first used the parameters  $h = 0.1, \Delta t = 0.025, \mu = 1, A_1 = 1, A_2 = 0.5, x_1 = 15, x_2 = 30$  through the interval  $0 \leq x \leq 80$  which is used by Zaki [5]. These parameters provide solitary waves of magnitudes 1 and 0.5

**Table 1. Invariants and error norms for single solitary waves.**

$t$	Linearization	$C_1$	$C_2$	$C_3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0		0.7853966	0.1666664	0.0052083	0.0000000	0.0000000
5		0.7853966	0.1666664	0.0052083	0.0000979	0.0000622
10	First	0.7853966	0.1666664	0.0052083	0.0002113	0.0001368
15		0.7853966	0.1666664	0.0052083	0.0003432	0.0002251
20		0.7853966	0.1666664	0.0052083	0.0004969	0.0003309
0		0.7853966	0.1666664	0.0052083	0.0000000	0.0000000
5		0.7853966	0.1666664	0.0052083	0.0000972	0.0000627
10	Second	0.7853966	0.1666664	0.0052083	0.0002102	0.00001378
15		0.7853966	0.1666664	0.0052083	0.0003419	0.0002272
20		0.7853966	0.1666664	0.0052083	0.0004957	0.0003331
0		0.7853966	0.1666664	0.0052083	0.0000000	0.0000000
5		0.7854325	0.1666908	0.0052098	0.0237333	0.0228190
10	Third	0.7854685	0.1667152	0.0052114	0.0480311	0.0454089
15		0.7855045	0.1667397	0.0052129	0.0734307	0.0678603
20		0.7855405	0.1667641	0.0052144	0.1004249	0.0900579
20 [5]		0.785397	0.166667	0.005210	0.0034500	0.0020300
20 [14]		0.7849545	0.1664765	0.0051995	0.2498925	0.2905166
20 [18]		0.7853977	0.1664735	0.0052083	0.2692812	0.2569972
20 [19]		-	-	-	0.1958878	0.1744330

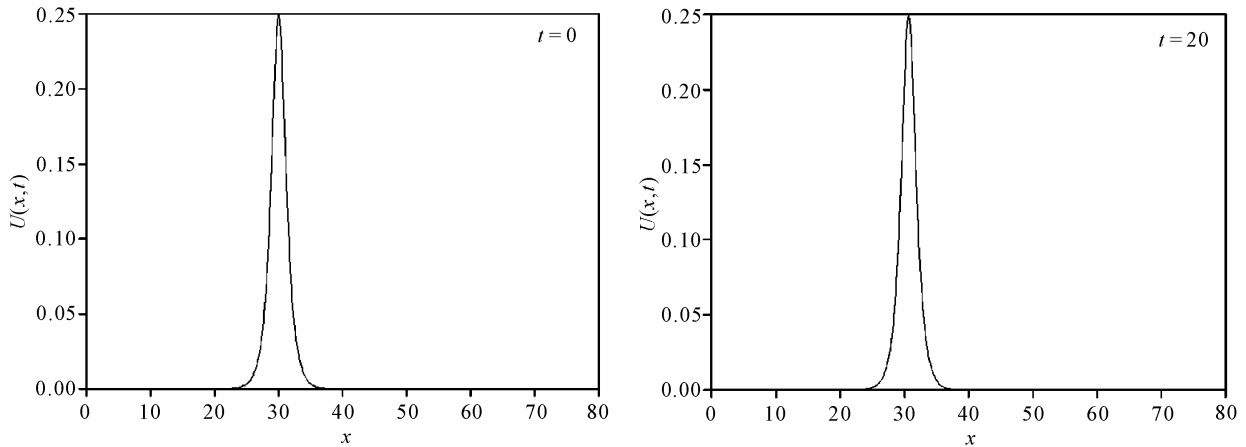


Figure 1. The motion of a single solitary wave.

and peak positions of them are located at  $x = 15$  and  $30$ . The analytical invariants are [14]

$$C_1 = \pi(A_1 + A_2) = 4.7123889,$$

$$C_2 = \frac{8}{3}(A_1^2 + A_2^2) = 3.3333333,$$

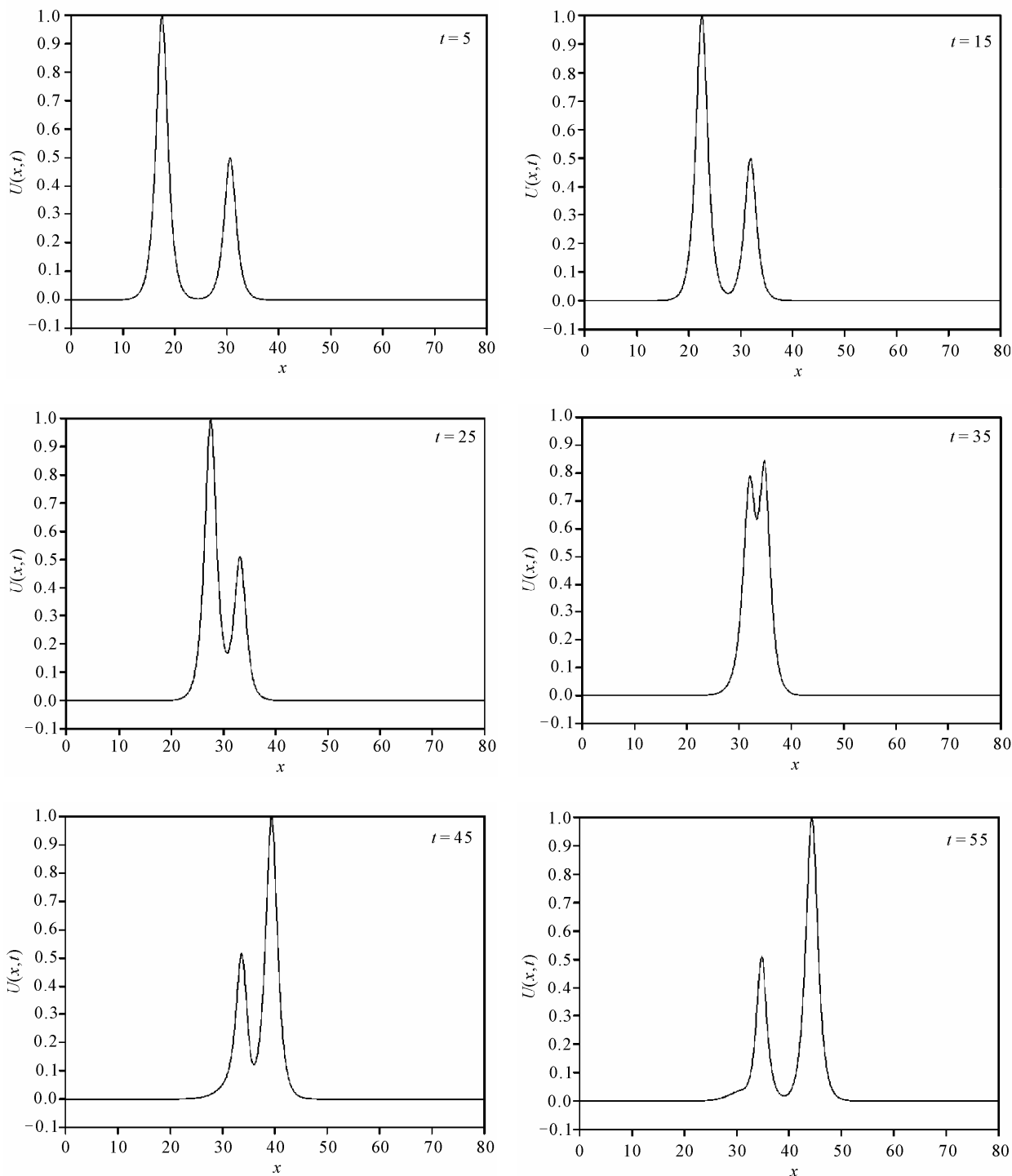
$$C_3 = \frac{4}{3}(A_1^4 + A_2^4) = 1.4166667.$$

Calculation is carried out with the time step  $\Delta t = 0.025$  and space step  $h = 0.1$  over the region  $0 \leq x \leq 80$ . The experiment was run from  $t = 0$  to  $t = 55$  to allow the interaction to take place. **Figure 2** shows the interaction of two positive solitary waves. It can be seen that at  $t = 5$  the wave with larger amplitude is on the left of the second wave with smaller amplitude. The larger wave catches up with the smaller one as time increases. Interaction started at about time  $t = 25$ , overlapping processes occurred between times  $t = 25$  and  $40$  and waves started to resume their original shapes after time  $t = 40$ . For the first and second linearization techniques at  $t = 55$ , the amplitude of larger waves is  $1.000149$  at the point  $x = 44.4$  whereas the amplitude of the smaller one is  $0.507317$  at the point  $x = 34.6$ . It is found that the absolute difference in amplitude is  $7.3 \times 10^{-3}$  for the smaller wave and  $0.149 \times 10^{-3}$  for the larger wave for this algorithm. For the third linearization technique at  $t = 55$ , the amplitude of larger waves is  $0.995933$  at the point  $x = 44.7$  whereas the amplitude of the smaller one is  $0.507477$  at the point  $x = 34.6$ . It is found that the absolute difference in amplitude is  $7.4 \times 10^{-3}$  for the smaller wave and  $4 \times 10^{-3}$  for the larger wave for this algorithm. In **Table 2** we compares values invariants of the two solitary waves with results from first, second and third linearization. We see from the **Table 2** that for the first and second linearization techniques, all 3 invariants are conserved by less than

$9.9 \times 10^{-5}$  during the experiment. Thus we have found that the conservation quantities are satisfactorily constant

Table 2. Invariants for interaction of two solitary wave.

$A_1 = 1, A_2 = 0.5$				
$t$	Linearization	$C_1$	$C_2$	$C_3$
0		4.7123733	3.3333294	1.4166643
5		4.7123660	3.3333183	1.4166532
15		4.7123494	3.3332959	1.4166308
25	First	4.7123331	3.3332741	1.4166083
35		4.7123243	3.3335335	1.4165818
45		4.7123127	3.3332470	1.4165824
55		4.7122960	3.3332247	1.4165605
0		4.7123733	3.3333294	1.4166643
5		4.7123696	3.3333160	1.4166509
15		4.7123602	3.3332892	1.4166241
25	Second	4.7123494	3.3332603	1.4165947
35		4.7123274	3.3332088	1.4165317
45		4.7123380	3.3332270	1.4165618
55		4.7123291	3.3332012	1.4165363
0		4.7123733	3.3333294	1.4166643
5		4.7318586	3.3857414	1.4689535
15		4.7738238	3.5004328	1.5869329
25	Third	4.8195659	3.6284390	1.7245060
35		4.8528439	3.7197153	1.8236340
45		4.9055278	3.8734565	2.0013048
55		4.9694524	4.0643034	2.2347812



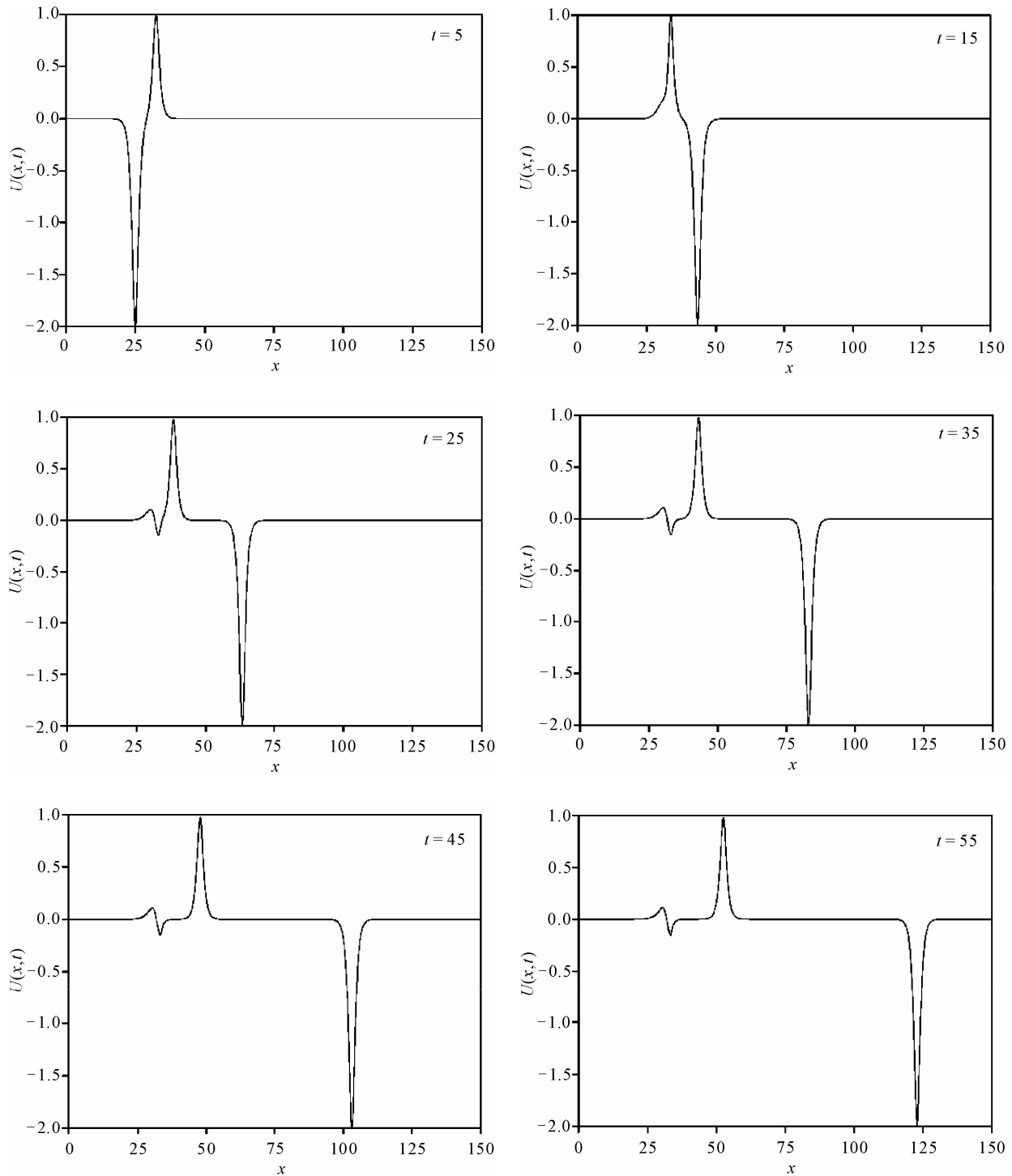
**Figure 2. Interaction of two solitary waves at different times.**

with the proposed algorithm.

We have also studied the interaction of two solitary waves with the following parameters:  $\mu = 1$ ,  $x_1 = 15$ ,  $x_2 = 30$ ,  $A_1 = -2$ ,  $A_2 = 1$ , together with time step  $\Delta t = 0.025$  and space step  $h = 0.1$  in the range  $0 \leq x \leq 150$ . The experiment was run from  $t = 0$  to  $t = 55$  to

allow the interaction to take place. **Figure 3** shows the development of the solitary wave interaction.

As is seen from the **Figure 3**, at  $t = 0$  a wave with the negative amplitude is on the left of another wave with the positive amplitude. The larger wave with the negative amplitude catches up with the smaller one with



**Figure 3.** Interaction of two solitary waves at different times.

the positive amplitude as the time increases. For the first linearization technique at  $t=55$ , the amplitude of the smaller wave is  $0.974695$  at the point  $x=52.5$ , whereas the amplitude of the larger one is  $-1.989036$  at the point  $x=122.8$ . For the second linearization technique at  $t=55$ , the amplitude of the smaller wave

is  $0.972778$  at the point  $x=52.5$ , whereas the amplitude of the larger one is  $-1.986701$  at the point  $x=122.8$ . It is found that the absolute difference in amplitudes is  $0.253 \times 10^{-1}$ ,  $0.272 \times 10^{-1}$  for the smaller wave and  $0.133 \times 10^{-1}$ ,  $0.109 \times 10^{-1}$  for the larger wave, respectively.



The analytical invariants by using Equation (1) can be found as  $C_1 = -3.1415927$ ,  $C_2 = 13.3333333$ ,  $C_3 = 22.6666667$ . **Table 3** lists the values of the invariants of the two solitary waves with amplitudes  $A_1 = -2$ ,  $A_2 = 1$ . It can be seen that the values obtained for the invariants are satisfactorily constant during the computer run.

### 8. The Maxwellian Initial Condition

As a last study, we consider here is the numerical solution of the Equation (1) with the Maxwellian initial condition

$$U(x, 0) = e^{-x^2} \tag{14}$$

with the boundary conditions

$$U(-20, t) = U_x(-20, t) = U(20, t) = U_x(20, t) = 0.$$

As it is known, Maxwellian initial condition (14) the behavior of the solution, depends on the values of  $\mu$ . So we have considered various values for  $\mu$ . For the first linearization technique the computations are carried out for the cases  $\mu = 1, 0.5, 0.1, 0.05, 0.02$  and  $0.005$  which are used in the earlier papers [5,14]. When  $\mu = 1, 0.5$  is used as shown **Figures 4(a)** and **(b)** at time  $t = 12$  the Maxwellian initial condition does not cause development into a clean solitary wave. However with smaller values of  $\mu = 0.1, 0.05, 0.02$  and  $0.005$  Max-

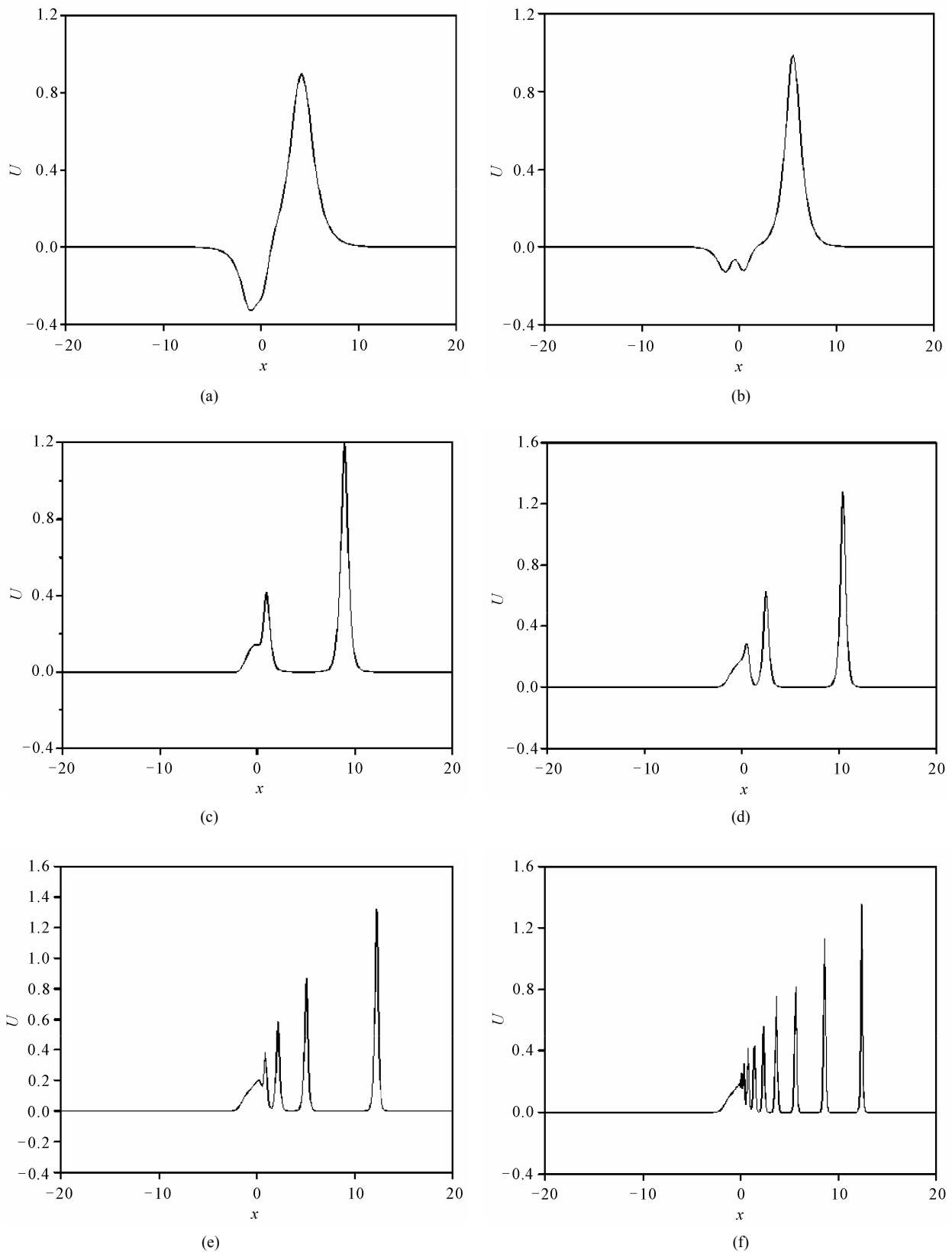
wellian initial condition breaks up into more solitary waves which were drawn in **Figures 4(c)-(f)** at time  $t = 12$ . The numerical conserved quantities with  $\mu = 1, 0.5, 0.1, 0.05, 0.02$  and  $0.005$  are given in **Table 4**. It is observed that the obtained values of the invariants remain almost constant during the computer run.

**Table 4. Invariants for Maxwellian initial condition, different  $\mu$ .**

$t$	$\mu$	$C_1$	$C_2$	$C_3$
0		1.772454	2.506607	0.886227
3		1.772972	2.506836	0.886561
6	1	1.775116	2.512628	0.890240
9		1.776365	2.514942	0.891795
12		1.776698	2.515136	0.891967
0		1.772454	1.879971	0.886227
3		1.772452	1.879970	0.886225
6	0.5	1.772451	1.879968	0.886224
9		1.772451	1.879967	0.886223
12		1.772450	1.879966	0.886222
0		1.772454	1.378646	0.886226
3		1.772420	1.378591	0.886174
6	0.1	1.772368	1.378507	0.886054
9		1.772316	1.378424	0.885933
12		1.772264	1.378340	0.885813
0		1.772454	1.315980	0.886227
3		1.772266	1.315654	0.885789
6	0.05	1.771976	1.315150	0.884947
9		1.771685	1.314648	0.884107
12		1.771396	1.314147	0.883270
0		1.772454	1.278380	0.886227
3		1.770834	1.275265	0.880933
6	0.02	1.768546	1.271572	0.874402
9		1.766186	1.266707	0.864400
12		1.763931	1.262351	0.855782
0		1.772454	1.259581	0.886227
3		1.757684	1.254254	0.928815
6	0.005	1.738212	1.227138	0.880527
9		1.722397	1.171836	0.714373
12		1.710490	1.162646	0.737751

**Table 3. Invariants for interaction of two solitary wave.**

$A_1 = -2, A_2 = 1$				
$t$	Linearization	$C_1$	$C_2$	$C_3$
0		-3.1415739	13.3332981	22.6665313
5		-3.1373324	13.3219118	22.6210653
15		-3.1227097	13.2806152	22.4483653
25	First	-3.1143337	13.2581754	22.3589494
35		-3.1060334	13.2359744	22.2706624
45		-3.0978106	13.2140028	22.1834701
55		-3.0896638	13.1922565	22.0973499
0		-3.1415739	13.3332981	22.6665313
5		-3.1391704	13.3196543	22.6120524
15		-3.1325076	13.2805526	22.4637733
25	Second	-3.1277388	13.2535359	22.3561477
35		-3.1230138	13.2268810	22.2501653
45		-3.1183416	13.2005566	22.1457584
55		-3.1137210	13.1745553	22.0428880



**Figure 4.** Maxwellian initial condition, state at time  $t = 12$  (a)  $\mu = 1$ , (b)  $\mu = 0.5$ , (c)  $\mu = 0.1$ , (d)  $\mu = 0.05$ , (e)  $\mu = 0.02$ , (f)  $\mu = 0.005$ .

## 9. Conclusions

In this study, a numerical solution of the MEW equation based on the septic B-spline finite element has been presented with three different linearization techniques. Three test problems are worked out to examine the performance of the algorithms. The performance and accuracy of the method were demonstrated by calculating the error norms  $L_2$  and  $L_\infty$  on the motion of a single solitary wave. For the first and second linearization techniques, the error norms are sufficiently small and the invariants are satisfactorily constant in all computer run. The obtained results from the first and the second linearization techniques are almost the same and the computed results show that the present method is a remarkably successful numerical technique for solving the MEW equation and can also be efficiently applied to other types of non-linear problem.

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