On the $m-$extension of Fibonacci $p-$functions with period k

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Abstract

Let $f_{p,m}$ be a real valued function on $\mathbb{R}$, p be nonnegative integer, k be a positive integer and m be a nonnegative real number. For all $x \in \mathbb{R}$, $f_{p,m}(x + (p + 1)k) = mf_{p,m}(x + pk) + f_{p,m}(x)$, we call this function $m-$extension of Fibonacci $p-$function with period k. In this paper, we present basic properties of $m-$extension of Fibonacci $p-$functions with period k. Specifying p and m, we obtain Fibonacci ($p = 1$, $m = 1$) and Pell ($p = 1$, $m = 2$) functions. Furthermore, we define $m-$extension of odd Fibonacci $p-$functions with period k. Moreover, we analyze some properties by using notion of $f-$even and $f-$odd functions with period k. We also demonstrate the products and quotients of these functions and provide new results in the development of Fibonacci functions with period k.

Keywords: $m-$extension of Fibonacci $p-$function with period k, $m-$extension of odd Fibonacci $p-$function with period k, $f-$even function with period k, $f-$odd functions with period k.
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1. Introduction

Fibonacci numbers is one of the most popular and fascinating linear sequences in mathematics and related fields. The classical Fibonacci sequence is defined by $F_{n+2} = F_{n+1} + F_n$, for $n \in \mathbb{N}$, with initial conditions $F_0 = 0, \quad F_1 = 1$. Up until now, many authors have studied the sums, representations, properties, relations with another mathematical topics, applications and generalizations of the Fibonacci sequence extensively (see [1–15]). Falcon introduced kth Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ that arises in the study of the recursive application of two geometrical transformations used in the well known four triangle longest edge (4TLE) partition[2]. In [7], Yazlik and Taskara defined generalized $k-$Horadam sequence and proved the properties of this sequence by means of determinant. Stakhov and Rozin presented, one of the important mathematical discoveries of the modern Golden Section and Fibonacci numbers theory, Fibonacci $p-$numbers and some properties of this sequence, $F_p(n) = F_p(n-1) + F_p(n-p-1)$, in [10]. Later on, the authors defined the

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$m-$extension of the Fibonacci $p-$numbers as

$$F_{p,m}(n+p+1) = mF_{p,m}(n+p) + F_{p,m}(n) \tag{1}$$

with initial conditions $F_{p,m}(0) = 0, F_{p,m}(1) = 1, F_{p,m}(2) = m, F_{p,m}(3) = m^2, \ldots, F_{p,m}(p+1) = m^p$, where $p, n \in \mathbb{N}$ and m is positive real number. For different values of p and m in equation (1), it can be reduced into different numerical sequences. For example, if $(p, m) = (1, 1)$, the Fibonacci sequence is obtained as $F_{n+2} = F_{n+1} + F_n$. If $(p, m) = (1, 2)$, the Pell sequence is obtained as $P_{n+2} = 2P_{n+1} + P_n$. If $p = 1$ and $m = k$, the k-Fibonacci sequence is obtained as $F_{k,n+2} = kF_{k,n+1} + F_{k,n}$ [9]. Recently, one of the important application of these integer sequences is continuous functions. Han et al.,[16], considered Fibonacci functions on the real numbers $\mathbb{R}$, i.e., functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x+2) = f(x+1)+f(x)$. Also they presented some properties of these functions by using the concept of $f-$even and $f-$odd functions. Moreover, they showed that if f is Fibonacci function then $\lim\limits_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. Afterwards, Sroysang extended Fibonacci functions to Fibonacci functions with period k as $f(x+2k) = f(x+k)+f(x)$ for all $x \in \mathbb{R}$ in [17]. In [18], Rabago defined the second order linear recurrent function with period k, $w(x+2k) = rw(x+k)+sw(x)$, where r, s are nonnegative real numbers, which is generalization of the Fibonacci function with period k.

Up until now, authors investigated some properties of the continuous functions of the second order linear recursive integer sequences. In this paper, we extend these properties to the continuous function in terms of $m-$extension of Fibonacci $p-$numbers which is defined by the $(p+1)th$ order linear recursive relation. We present some properties of the $m-$extension of Fibonacci $p-$functions with period k using the concept of $f-$even and $f-$odd functions with period k. We also define $m-$extension of odd Fibonacci $p-$functions with period k, investigate the product and the limit of $m-$extension of Fibonacci $p-$functions with period k.

2. $m-$extension of Fibonacci $p-$functions with period k

In this section we define $m-$extension of Fibonacci $p-$functions with period k and present some properties of these functions.

Definition 2.1. *Let k be a positive integer, p be nonnegative integer and m be a nonnegative real number. A function $f_{p,m} : \mathbb{R} \to \mathbb{R}$ is called an $m-$extension of Fibonacci $p-$function with period k if it satisfies the equation*

$$f_{p,m}(x+(p+1)k) = mf_{p,m}(x+pk) + f_{p,m}(x), \quad \forall x \in \mathbb{R}. \tag{2}$$

Taking $(p, m) = (1, 1)$ and $(p, m) = (1, 2)$ in (2), we obtain Fibonacci and Pell function with period k, respectively (see [17, 18]).

2

Example 2.1. Let α be the positive real number that satisfies the equation $\alpha^{p+1} = m\alpha^p + 1$, k be a positive integer, p be a nonnegative integer. Then, $f_{p,m}(x) = \alpha^{\frac{x}{k}}$ is an $m-$extension of Fibonacci $p-$function with period k.

The following are special cases of the previous example:

1. If $(p, m) = (1, 1)$ then the function $f_{1,1}(x) = \phi^{\frac{x}{k}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is known as golden ratio, is an example of $m-$extension of Fibonacci $p-$function with period k in [17].
2. If $(p, m) = (1, 2)$ then the function $f_{1,2}(x) = \sigma^{\frac{x}{k}}$, where $\sigma = 1 + \sqrt{2}$ is known as silver ratio, is an example of $m-$extension of Pell $p-$function with period k in [18].

Proposition 2.1. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of Fibonacci $p-$function with period k. Assume that $f_{p,m}$ is s times differentiable. Then $\{f'_{p,m}, f''_{p,m}, \ldots, f^{(s)}_{p,m}\}$ are also $m-$extension of odd Fibonacci $p-$functions with period k.*

Proposition 2.2. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of Fibonacci $p-$function with period k. Define $g_t(x) = f_{p,m}(x + t)$, for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then, $g_t(x)$ is also an $m-$extension of Fibonacci $p-$function with period k.*

Proof. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned} g_t(x + (p + 1)k) &= f_{p,m}(x + (p + 1)k + t) \\ &= mf_{p,m}(x + pk + t) + f_{p,m}(x + t) \\ &= mg_t(x + pk) + g_t(x) \end{aligned}$$

is an $m-$extension of Fibonacci $p-$function with period k. □

Example 2.2. Let p be a nonnegative integer, k be positive integer and $t \in \mathbb{R}$. Define $g_t : \mathbb{R} \to \mathbb{R}$ by

$$g_t(x) = \alpha^{\frac{x+t}{k}}, \qquad \forall x \in \mathbb{R}, \tag{3}$$

then $g_t(x)$ is an $m-$extension of Fibonacci $p-$function with period k.

As special cases of the previous example, we have

1. If $(p, m) = (1, 1)$, then the function $g_t(x) = f_{1,1}(x + t) = \phi^{\frac{x+t}{k}}$ is an example of $m-$extension of Fibonacci $p-$function with period k in [17].
2. If $(p, m) = (1, 2)$, then the function $g_t(x) = f_{1,2}(x + t) = \sigma^{\frac{x}{k}}$ is an example of $m-$extension of Pell $p-$function with period k in [18].

Theorem 2.1. *Let $f_{p,m}$ be an $m-$extension of Fibonacci $p-$function with period k and $F_{p,m}$ be an $m-$extension of Fibonacci $p-$sequence with the initial conditions $F_{p,m}(0) = 0, F_{p,m}(1) = 1, F_{p,m}(2) = m, \ldots, F_{p,m}(p) = m^{p-1}$. Then, for $n \geq 2p$ and $\forall x \in \mathbb{R}$,*

$$f_{p,m}(x + nk) = F_{p,m}(n - p + 1)f(x + pk) + \sum_{i=0}^{p-1} F_{p,m}(n - p - i)f(x + ik). \tag{4}$$

Proof. We prove the theorem by induction on n. For $n = 2p$, we get

$$\begin{aligned}
f_{p,m}(x + 2pk) &= mf_{p,m}(x + (2p - 1)k) + f_{p,m}(x + (p - 1)k) \\
&= m\Big[mf_{p,m}(x + (2p - 2)k) + f_{p,m}(x + (p - 2)k)\Big] \\
&\quad +f_{p,m}(x + (p - 1)k) \\
&= m^2 f_{p,m}(x + (2p - 2)k) + f_{p,m}(x + (p - 1)k) \\
&\quad +mf_{p,m}(x + (p - 2)k) \\
&= m^3 f_{p,m}(x + (2p - 3)k) + f_{p,m}(x + (p - 1)k) \\
&\quad +mf_{p,m}(x + (p - 2)k) + m^2 f_{p,m}(x + (p - 3)k).
\end{aligned}$$

Continuing this process $(p - 3)$ times, we have

$$\begin{aligned}
f_{p,m}(x + 2pk) &= m^p f_{p,m}(x + pk) + f_{p,m}(x + (p - 1)k) \\
&\quad +mf_{p,m}(x + (p - 2)k) + \cdots + m^{p-1} f_{p,m}(x).
\end{aligned}$$

By considering the initial conditions of the $m-$extension of Fibonacci $p-$sequence, we obtain

$$\begin{aligned}
f_{p,m}(x + 2pk) &= F_{p,m}(p + 1)f_{p,m}(x + pk) + F_{p,m}(1)f_{p,m}(x + (p - 1)k) \\
&\quad +F_{p,m}(2)f_{p,m}(x + (p - 2)k) + \cdots + F_{p,m}(p - 1)f_{p,m}(x + k) \\
&\quad +F_{p,m}(p)f_{p,m}(x).
\end{aligned}$$

Assume that equation (4) is true for $n \geq 2p+1$. Then we write

$$\begin{aligned}
f_{p,m}(x+(n+1)k) &= mf_{p,m}(x+nk)+f_{p,m}(x+(n-p)k) \\
&= m\Big[F_{p,m}(n-p+1)f_{p,m}(x+pk) \\
&\quad +F_{p,m}(n-2p+1)f_{p,m}(x+(p-1)k) \\
&\quad +\cdots+F_{p,m}(n-p)f_{p,m}(x)\Big] \\
&\quad +F_{p,m}(n-2p+1)f_{p,m}(x+pk) \\
&\quad +F_{p,m}(n-3p+1)f_{p,m}(x+(p-1)k) \\
&\quad +\cdots+F_{p,m}(n-2p)f_{p,m}(x) \\
&= (mF_{p,m}(n-p+1)+F_{p,m}(n-2p+1))f_{p,m}(x+pk) \\
&\quad +(mF_{p,m}(n-2p+1)+F_{p,m}(n-3p+1))f_{p,m}(x+(p-1)k) \\
&\quad +\cdots+(mF_{p,m}(n-p)+F_{p,m}(n-2p))f_{p,m}(x) \\
&= F_{p,m}(n-p+2)f(x+pk)+\sum_{i=0}^{p-1}F_{p,m}(n+1-p-i)f(x+ik),
\end{aligned}$$

which completes the proof. □

Corollary 2.1. *Let $f_{p,m}$ be an $m-$extension of Fibonacci $p-$function with period k and $F_{p,m}$ be the sequence of $m-$extension of Fibonacci $p-$numbers. Then, for any $x \in \mathbb{R}$ and $n \geq 2p$,*

$$\alpha^n = F_{p,m}(n-p+1)\alpha^p + \sum_{i=0}^{p-1} F_{p,m}(n-p-i)\alpha^i. \tag{5}$$

Proof. From example (2.1), we say that $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, k is a positive integer, is an $m-$extension of Fibonacci $p-$function with period k, so it satisfies the Equation(2), for all $x \in \mathbb{R}$, i.e.

$$\begin{aligned}
\alpha^{\frac{x+nk}{k}} &= f_{p,m}(x+nk) \\
&= F_{p,m}(n-p+1)f(x+pk)+\sum_{i=0}^{p-1}F_{p,m}(n-p-i)f(x+ik) \\
&= \alpha^{\frac{x}{k}+p}F_{p,m}(n-p+1)+\alpha^{\frac{x}{k}}F_{p,m}(n-p)+\alpha^{\frac{x}{k}+1}F_{p,m}(n-p-1) \\
&+ \alpha^{\frac{x}{k}+2}F_{p,m}(n-p-2)+\cdots+\alpha^{\frac{x}{k}+p-1}F_{p,m}(n-2p+1).
\end{aligned}$$

Upon simplifying, we get

$$\alpha^n = F_{p,m}(n-p+1)\alpha^p + \sum_{i=0}^{p-1} F_{p,m}(n-p-i)\alpha^i, \tag{6}$$

which is desired. □

3. $m-$extension of odd Fibonacci $p-$functions with period k

In this section, we present the $m-$extension of odd Fibonacci $p-$function with period k and analyze some properties of these functions.

Definition 3.1. *Let p be a nonnegative integer, m be a nonnegative real number and k be a positive integer. A function $f_{p,m} : \mathbb{R} \to \mathbb{R}$ is said to be $m-$extension of odd Fibonacci $p-$function with period k, if $f_{p,m}$ satisfies*

$$f_{p,m}(x + (p + 1)k) = -mf_{p,m}(x + pk) + f_{p,m}(x), \quad \forall x \in \mathbb{R}. \tag{7}$$

Example 3.1. Let α be the positive real number that satisfies the equation $\alpha^{p+1} = m\alpha^p + 1$, k be a positive integer, p be a nonnegative integer. Therefore $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, for all $x \in \mathbb{R}$, is an $m-$extension of odd Fibonacci $p-$function with period k.

Proposition 3.1. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of odd Fibonacci $p-$function with period k. Assume that $f_{p,m}$ is s times differentiable. Then $\{f'_{p,m}, f''_{p,m}, \ldots, f^{(s)}_{p,m}\}$ are also $m-$extension of odd Fibonacci $p-$functions with period k.*

Proposition 3.2. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of odd Fibonacci $p-$function with period k. Define $g_t(x) = f_{p,m}(x + t)$, for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then, g_t is also an $m-$extension of odd Fibonacci $p-$function with period k.*

Proof. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned} g_t(x + (p + 1)k) &= f_{p,m}(x + (p + 1)k + t) \\ &= -mf_{p,m}(x + pk + t) + f_{p,m}(x + t) \\ &= -mg_t(x + pk) + g_t(x). \end{aligned}$$

Therefore, $g_t(x)$ is an $m-$extension of odd Fibonacci $p-$function with period k. □

4. Products of $m-$extension of Fibonacci $p-$functions with period k

In this section, we present the product of $m-$extension of Fibonacci $p-$functions with period k by using the concept of $f-$even and $f-$odd functions with period k which are defined in [16].

Definition 4.1 ([16]). *Let $k \in \mathbb{N}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be such that if $\varphi h \equiv 0$ where $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h \equiv 0$. The map φ is said to be an $f-$even and $f-$odd function with period k if $\varphi(x + k) = \varphi(x)$ and if $\varphi(x + k) = -\varphi(x)$, respectively, for any $x \in \mathbb{R}$.*

Theorem 4.1. *Let k be a positive integer, $\varphi : \mathbb{R} \to \mathbb{R}$ be an $f-$even function with period k and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k if and only if $(\varphi f_{p,m})$ is an $m-$extension of Fibonacci $p-$function with period k.*

Proof. First, we assume that $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k. For any $x \in \mathbb{R}$,

$$\begin{aligned} (\varphi f_{p,m})(x+(p+1)k) &= \varphi(x+(p+1)k) f_{p,m}(x+(p+1)k) \\ &= \varphi(x+(p+1)k)\Big[m f_{p,m}(x+pk) + f_{p,m}(x)\Big] \\ &= m\varphi(x+pk) f_{p,m}(x+pk) + \varphi(x) f_{p,m}(x) \\ &= m(\varphi f_{p,m})(x+pk) + (\varphi f_{p,m})(x). \end{aligned}$$

Therefore, $(\varphi f_{p,m})$ is an $m-$extension of Fibonacci $p-$function with period k. Next, assume that $(\varphi f_{p,m})$ is an $m-$extension of Fibonacci $p-$function with period k, then

$$\begin{aligned} \varphi(x+k) f_{p,m}(x+(p+1)k) &= \varphi(x+(p+1)k) f_{p,m}(x+(p+1)k) \\ &= (\varphi f_{p,m})(x+(p+1)k) \\ &= m(\varphi f_{p,m})(x+pk) + (\varphi f_{p,m})(x) \\ &= m\varphi(x+pk) f_{p,m}(x+pk) + \varphi(x) f_{p,m}(x) \\ &= \varphi(x+k)\Big[m f_{p,m}(x+pk) + f_{p,m}(x)\Big]. \end{aligned}$$

Thus, $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k. This completes the proof. □

Example 4.1. Let k be a positive integer and define $\gamma(x) = x - \lfloor x \rfloor$ which is an example of $f-$even function. Moreover, recall that the function $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, where α is positive real root of the characteristic equation $\alpha^{p+1} - m\alpha^p - 1 = 0$, is an $m-$extension of Fibonacci $p-$function with period k. By using Theorem 4.1, for all $x \in \mathbb{R}$

$$(\gamma f_{p,m})(x) = (x - \lfloor x \rfloor)\alpha^{\frac{x}{k}} \tag{8}$$

is an example of an $m-$extension of Fibonacci $p-$function with period k.

Theorem 4.2. *Let k be a positive integer, $\varphi : \mathbb{R} \to \mathbb{R}$ be an $f-$even function with period k and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, $f_{p,m}$ is an $m-$extension of odd Fibonacci $p-$function with period k if and only if $(\varphi f_{p,m})$ is an $m-$extension of odd Fibonacci $p-$function with period k.*

Proof. First, assume that $f_{p,m}$ is an $m-$extension of odd Fibonacci $p-$function with period k, for any $x \in \mathbb{R}$

$$\begin{aligned} (\varphi f_{p,m})(x+(p+1)k) &= \varphi(x+(p+1)k) f_{p,m}(x+(p+1)k) \\ &= \varphi(x+(p+1)k)\left[-m f_{p,m}(x+pk) + f_{p,m}(x)\right] \\ &= -m\varphi(x+pk) f_{p,m}(x+pk) + \varphi(x) f_{p,m}(x) \\ &= -m(\varphi f_{p,m})(x+pk) + (\varphi f_{p,m})(x). \end{aligned}$$

Therefore, $(\varphi f_{p,m})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that $(\varphi f_{p,m})$ is an $m-$extension of odd Fibonacci $p-$function with period k, for any $x \in \mathbb{R}$, then

$$\begin{aligned}
\varphi(x+k) f_{p,m}(x+(p+1)k) &= \varphi(x+(p+1)k) f_{p,m}(x+(p+1)k) \\
&= (\varphi f_{p,m})(x+(p+1)k) \\
&= -m(\varphi f_{p,m})(x+pk) + (\varphi f_{p,m})(x) \\
&= -m\varphi(x+pk) f_{p,m}(x+pk) + \varphi(x) f_{p,m}(x) \\
&= \varphi(x+k)\left[-m f_{p,m}(x+pk) + f_{p,m}(x)\right].
\end{aligned}$$

Thus, $f_{p,m}$ is an $m-$extension of odd Fibonacci $p-$function with period k. This completes the proof. □

Example 4.2. Let k be a positive integer and define $\gamma(x) = x - \lfloor x \rfloor$ which is an example of $f-$even function [16]. Moreover, recall that the function $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, where α is positive real root of the characteristic equation $\alpha^{p+1} + m\alpha^p - 1 = 0$, is an $m-$extension of odd Fibonacci $p-$function with period k. By using Theorem (4.2), for all $x \in \mathbb{R}$

$$(\gamma f_{p,m})(x) = (x - \lfloor x \rfloor)\alpha^{\frac{x}{k}} \tag{9}$$

is an example of an $m-$extension of odd Fibonacci $p-$function with period k.

Theorem 4.3. *Let k be a positive integer, f_{p,m_1} and f_{p,m_2} be two $m-$extension of Fibonacci $p-$functions with period k satisfying*

$$f_{p,m_1}(x+(p+1)k) = m_1 f_{p,m_1}(x+pk) + f_{p,m_1}(x), \quad \forall x \in \mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = m_2 f_{p,m_2}(x+pk) + f_{p,m_2}(x), \quad \forall x \in \mathbb{R},$$

where m_1, m_2 are nonnegative real numbers. Suppose that the following conditions are satisfied:

(C1) f_{p,m_1} is an $f-$even function,

(C2) f_{p,m_2} is an $f-$odd function,

(C3) if p is odd then $m_1 = m_2$,

(C4) if p is even then $m_1 = -m_2$,

(C5) $\mu = m_1.m_2$.

Then $(f_{p,m_1} f_{p,m_2})(x)$ is also an $m-$extension of Fibonacci $p-$function with period k.

Proof. Assume that f_{p,m_1} and f_{p,m_2} be two $m-$extension of Fibonacci $p-$functions with period k and the conditions *(C1)*,*(C2)*,*(C3)*,*(C4)* and *(C5)* are satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= [m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x)]\\ &\quad [m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x)]\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)+m_1f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad +m_2f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= \mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x), \quad \forall x\in\mathbb{R}.\end{aligned}$$

Thus, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of Fibonacci $p-$function with period k. □

Theorem 4.4. *Let k be a positive integer, f_{p,m_1} be an $m-$extension of Fibonacci $p-$function with period k and f_{p,m_2} be an $m-$extension of odd Fibonacci $p-$function with period k satisfying*

$$f_{p,m_1}(x+(p+1)k)=m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x), \quad \forall x\in\mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k)=-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x), \quad \forall x\in\mathbb{R},$$

where m_1,m_2 are nonnegative real numbers. Suppose that (C6), (C9) and one the following conditions (C7) and (C8) are satisfied:

(C6) if p is odd or even then $m_1=m_2$,

(C7) f_{p,m_1} and f_{p,m_2} are both $f-$even functions,

(C8) f_{p,m_1} and f_{p,m_2} are both $f-$odd functions,

(C9) $\mu=m_1m_2$

Then, $(f_{p,m_1}f_{p,m_2})$ is also an $m-$extension of odd Fibonacci $p-$function with period k.

Proof. First assume that f_{p,m_1} is an $m-$extension of Fibonacci $p-$function with period k and f_{p,m_2} is an $m-$extension of odd Fibonacci $p-$function with period k and the conditions *(C6)*, *(C9)* and *(C7)* are satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= \Big[m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\ &= -m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)+m_1 f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad -m_2 f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= -m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= -\mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),\end{aligned}$$

$\forall x\in\mathbb{R}$. Therefore, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that f_{p,m_1} is an $m-$extension of Fibonacci $p-$function with period k and f_{p,m_2} is an $m-$extension of odd Fibonacci $p-$function with period k and the conditions *(C6)*, *(C9)* and *(C8)* are satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= \Big[m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\ &= -m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)-m_1 f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad +m_2 f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= -m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= -\mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),\end{aligned}$$

$\forall x\in\mathbb{R}$. Thus, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. This proves the theorem. □

Theorem 4.5. *Let k be a positive integer, f_{p,m_1} and f_{p,m_2} be two $m-$extension of Fibonacci $p-$functions with period k satisfying*

$$f_{p,m_1}(x+(p+1)k)=m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x),\quad \forall x\in\mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k)=m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x),\quad \forall x\in\mathbb{R},$$

where m_1,m_2 are nonnegative real numbers. Suppose that (C10),(C11) and one the conditions (C7) and (C8) are satisfied:

(C10) if p is odd or even then $m_1=-m_2$,

(C11) $\mu = -m_1.m_2$.

Then, $(f_{p,m_1} f_{p,m_2})$ *is an* $m-$*extension of odd Fibonacci* $p-$*function with period* k.

Proof. First assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of Fibonacci $p-$functions with period k and the conditions *(C7)*, *(C10)* and *(C11)* are satisfied. Then,

$$\begin{aligned}
(f_{p,m_1} f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k) f_{p,m_2}(x+(p+1)k) \\
&= \Big[m_1 f_{p,m_1}(x+pk) + f_{p,m_1}(x)\Big]\Big[m_2 f_{p,m_2}(x+pk) + f_{p,m_2}(x)\Big] \\
&= m_1 m_2 f_{p,m_1}(x+pk) f_{p,m_2}(x+pk) \\
&\quad + f_{p,m_1}(x) f_{p,m_2}(x) + m_1 f_{p,m_1}(x+pk) f_{p,m_2}(x) \\
&\quad + m_2 f_{p,m_2}(x+pk) f_{p,m_1}(x) \\
&= m_1 m_2 f_{p,m_1}(x+pk) f_{p,m_2}(x+pk) \\
&\quad + f_{p,m_1}(x) f_{p,m_2}(x) \\
&= -\mu (f_{p,m_1} f_{p,m_2})(x+pk) + (f_{p,m_1} f_{p,m_2})(x),
\end{aligned}$$

$\forall x \in \mathbb{R}$. Therefore, $(f_{p,m_1} f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of Fibonacci $p-$functions with period k and the conditions *(C8)*, *(C10)* and *(C11)* are satisfied. Then the same result can be obtained. Therefore, $(f_{p,m_1} f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. □

Theorem 4.6. *Let* k *be a positive integer,* f_{p,m_1} *and* f_{p,m_2} *be two* $m-$*extension of odd Fibonacci* $p-$*functions with period* k *satisfying*

$$f_{p,m_1}(x+(p+1)k) = -m_1 f_{p,m_1}(x+pk) + f_{p,m_1}(x), \quad \forall x \in \mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = -m_2 f_{p,m_2}(x+pk) + f_{p,m_2}(x), \quad \forall x \in \mathbb{R},$$

where m_1, m_2 *are nonnegative real numbers. Suppose that the conditions (C1),(C2),(C3),(C4) and (C5) are satisfied. Then* $(f_{p,m_1} f_{p,m_2})(x)$ *is an* $m-$*extension of Fibonacci* $p-$*function with period* k.

Proof. Assume that f_{p,m_1} and f_{p,m_2} be two $m-$extension of odd Fibonacci $p-$functions with period k and the conditions *(C1),(C2),(C3),(C4)* and *(C5)* are satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= \Big[-m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)-m_1f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad -m_2f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= \mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x), \quad \forall x\in\mathbb{R}.\end{aligned}$$

Thus, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of Fibonacci $p-$function with period k. □

Theorem 4.7. *Let k be a positive integer, f_{p,m_1} and f_{p,m_2} be two $m-$extension of odd Fibonacci $p-$functions with period k satisfying*

$$f_{p,m_1}(x+(p+1)k) = -m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x), \quad \forall x\in\mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = -m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x), \quad \forall x\in\mathbb{R},$$

where m_1, m_2 are nonnegative real numbers. Suppose that (C10),(C11) and one the conditions (C7) and (C8) are satisfied. Then, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k.

Proof. First assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of odd Fibonacci $p-$functions with period k and the conditions *(C7)*, *(C10)* and *(C11)* are satisfied. Then,

$$\begin{aligned}(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\ &= \Big[-m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)-m_1f_{p,m_1}(x+pk)f_{p,m_2}(x)\\ &\quad -m_2f_{p,m_2}(x+pk)f_{p,m_1}(x)\\ &= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\ &\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\ &= -\mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),\end{aligned}$$

$\forall x\in\mathbb{R}$. Therefore, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of odd Fibonacci $p-$functions with period k and the conditions *(C8)*, *(C10)* and *(C11)* are satisfied. Then the same result can be obtained. Therefore, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. □

12

5. Quotients of $m-$extension of Fibonacci $p-$functions with period k

In this section, we discuss the limit of quotients of $m-$extension of Fibonacci $p-$functions with period k.

Theorem 5.1. *If $f_{p,m}$ is an $m-$extension of Fibonacci $p-$functions with period k, then the limit of the quotient $\frac{f_{p,m}(x+k)}{f_{p,m}(x)}$ exists.*

Proof. Let $k \in \mathbb{N}$, $m \in \mathbb{R}^+$, p be a nonnegative integer and $n \geq 2p$. Consider the quotient $Q(x) = \frac{f_{p,m}(x+k)}{f_{p,m}(x)}$, where $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k. We have two possibilities such that either $Q(x) < 0$ or $Q(x) > 0$. First, suppose that $Q(x) < 0$ then without loss of generality, $f_{p,m}(x) > 0$ and $f_{p,m}(x+k) < 0$. Therefore,

$$\begin{aligned}
f_{p,m}(x+2pk) &= mf_{p,m}(x+(2p-1)k) + f_{p,m}(x+(p-1)k) \\
&= m^2 f_{p,m}(x+(2p-2)k) + f_{p,m}(x+(p-1)k) \\
&\quad + mf_{p,m}(x+(p-2)k) \\
&= m^3 f_{p,m}(x+(2p-3)k) + f_{p,m}(x+(p-1)k) \\
&\quad + mf_{p,m}(x+(p-2)k) + m^2 f_{p,m}(x+(p-3)k) \\
&\vdots \\
&= m^p f_{p,m}(x+pk) + f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - m^{p-2} f_{p,m}(x+k) + m^{p-1} f_{p,m}(x) \\
&= F_{p,m}(p+1) f_{p,m}(x+pk) + F_{p,m}(1) f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - F_{p,m}(p-1) f_{p,m}(x+k) + F_{p,m}(p) f_{p,m}(x),
\end{aligned}$$

$$\begin{aligned}
f_{p,m}(x+(2p+1)k) &= mf_{p,m}(x+2pk) + f_{p,m}(x+pk) \\
&= m\Big[m^p f_{p,m}(x+pk) + f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - m^{p-2} f_{p,m}(x+k) + m^{p-1} f_{p,m}(x)\Big] + f(x+pk) \\
&= (m^{p+1}+1) f_{p,m}(x+pk) + mf_{p,m}(x+(p-1)k) \\
&\quad + \cdots - m^{p-1} f_{p,m}(x+k) + m^p f_{p,m}(x) \\
&= F_{p,m}(p+2) f_{p,m}(x+pk) + F_{p,m}(2) f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - F_{p,m}(p) f_{p,m}(x+k) + F_{p,m}(p+1) f_{p,m}(x).
\end{aligned}$$

13

$$\begin{aligned}
f_{p,m}(x+(2p+2)k) &= mf_{p,m}(x+(2p+1)k)+f_{p,m}(x+(p+1)k) \\
&= m\Big[(m^{p+1}+1)f_{p,m}(x+pk)+mf_{p,m}(x+(p-1)k) \\
&\quad +\cdots-m^{p-1}f_{p,m}(x+k)+m^{p}f_{p,m}(x)\Big] \\
&\quad +mf(x+pk)+f(x) \\
&= (m^{p+2}+2m)f_{p,m}(x+pk)+m^{2}f_{p,m}(x+(p-1)k) \\
&\quad +\cdots-m^{p}f_{p,m}(x+k)+(m^{p+1}+1)f_{p,m}(x) \\
&= F_{p,m}(p+3)f_{p,m}(x+pk)+F_{p,m}(3)f_{p,m}(x+(p-1)k) \\
&\quad +\cdots-F_{p,m}(p+1)f_{p,m}(x+k)+F_{p,m}(p+2)f_{p,m}(x).
\end{aligned}$$

Continuing this process, we have

$$\begin{aligned}
f_{p,m}(x+nk) &= F_{p,m}(n-p+1)f_{p,m}(x+pk) \\
&= +F_{p,m}(n-2p+1)f_{p,m}(x+(p-1)k) \\
&= +\cdots-F_{p,m}(n-p-1)f_{p,m}(x+k)+F_{p,m}(n-p)f_{p,m}(x)
\end{aligned}$$

and

$$\begin{aligned}
f_{p,m}(x+(n+1)k) &= F_{p,m}(n-p+2)f_{p,m}(x+pk) \\
&= +F_{p,m}(n-2p+2)f_{p,m}(x+(p-1)k) \\
&= +\cdots-F_{p,m}(n-p)f_{p,m}(x+k)+F_{p,m}(n-p+1)f_{p,m}(x),
\end{aligned}$$

where $F_{p,m}$ is an $m-$extension of Fibonacci $p-$sequence with the initial conditions, $F_{p,m}(0)=0$, $F_{p,m}(1)=1$, $F_{p,m}(2)=m$, $\ldots$, $F_{p,m}(p)=m^{p-1}$. Given $x'\in\mathbb{R}$, there exists $x\in\mathbb{R}$ such that $x'=x+nk$. Therefore,

$$\begin{aligned}
\frac{f_{p,m}(x'+k)}{f_{p,m}(x')} &= \frac{f_{p,m}(x+(n+1)k)}{f_{p,m}(x+nk)} \\
&= \left(\frac{F_{p,m}(n-p+2)f_{p,m}(x+pk)+\cdots-F_{p,m}(n-p)f_{p,m}(x+k)+F_{p,m}(n-p+1)f_{p,m}(x)}{F_{p,m}(n-p+1)f_{p,m}(x+pk)+\cdots-F_{p,m}(n-p-1)f_{p,m}(x+k)+F_{p,m}(n-p)f_{p,m}(x)}\right) \\
&= \left(\frac{F_{p,m}(n-p+2)\left[f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+2)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p+1)}{F_{p,m}(n-p+2)}f_{p,m}(x)\right]}{F_{p,m}(n-p+1)\left[f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)\right]}\right).
\end{aligned}$$

14

$$\lim_{n\to\infty}\left(\frac{f_{p,m}(x'+k)}{f_{p,m}(x')}\right) = \lim_{n\to\infty}\left(\frac{F_{p,m}(n-p+2)}{F_{p,m}(n-p+1)}\right)\left(\frac{f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+2)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p+1)}{F_{p,m}(n-p+2)}f_{p,m}(x)}{f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)}\right)$$

$$= \left(\lim_{n\to\infty}\frac{F_{p,m}(n-p+2)}{F_{p,m}(n-p+1)}\right)\left(\frac{f_{p,m}(x+pk)+\cdots-\lim_{n\to\infty}\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+2)}f_{p,m}(x+k)+\lim_{n\to\infty}\frac{F_{p,m}(n-p+1)}{F_{p,m}(n-p+2)}f_{p,m}(x)}{f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)}\right).$$

Let $N=n+1$. If $n\to\infty$ then $N\to\infty$. So, we can write the above expression as

$$\left(\lim_{n\to\infty}\frac{f_{p,m}(x'+k)}{f_{p,m}(x')}\right) = \lim_{n\to\infty}\left(\frac{F_{p,m}(n-p+2)}{F_{p,m}(n-p+1)}\right)\left(\frac{f_{p,m}(x+pk)+\cdots-\lim_{N\to\infty}\frac{F_{p,m}(N-p-1)}{F_{p,m}(N-p+1)}f_{p,m}(x+k)+\lim_{N\to\infty}\frac{F_{p,m}(N-p)}{F_{p,m}(N-p+1)}f_{p,m}(x)}{f_{p,m}(x+pk)+\cdots-\lim_{n\to\infty}\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\lim_{n\to\infty}\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)}\right)$$

$$= \alpha_m\left(\frac{f_{p,m}(x+pk)+\cdots-\lim_{N\to\infty}\frac{F_{p,m}(N-p-1)}{F_{p,m}(N-p+1)}f_{p,m}(x+k)+\lim_{N\to\infty}\frac{F_{p,m}(N-p)}{F_{p,m}(N-p+1)}f_{p,m}(x)}{f_{p,m}(x+pk)+\cdots-\lim_{n\to\infty}\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\lim_{n\to\infty}\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)}\right) = \alpha_m.$$

Here α_m is the unique positive real root of the characteristic equation of $m-$extension of Fibonacci $p-$sequence. Next, suppose that $Q(x)>0$, without loss of generality we assume $f_{p,m}(x)>0$, $f_{p,m}(x+k)>0$. Identically, we can easily obtain that $\lim_{n\to\infty}\left(\frac{f_{p,m}(x+(n+1)k)}{f_{p,m}(x+nk)}\right)=\alpha_m$. Hence we omit the proof. □

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