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Some base spaces and core theorems of new type

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Abstract

In this paper, we constructed two new base sequence spaces, denoted rf and rf_0, and we investigated some of their important properties. Then, by using matrix domains, we defined other sequence spaces on these base spaces, called zrf and zrf_0. Finally, we introduced the$B_{\hat{R}} - core$ of a complex-valued sequence and we examined some inclusion theorems related to this new type of core.

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1. Preliminaries, background and notations

There are several methods by which one can construct new spaces from a given space. The easiest one is to derive a linear subspace from any given space. For example, let us suppose that V is the set of functions defined as $e^{i\xi t} = e_\xi(t)$ in the complex space $\mathbb{C}(-\infty,\infty)$ with the norm $\|x\| = \sup_{(-\infty,\infty)} |x(t)|$ and $\xi \in \mathbb{R}$, the set of real numbers. In this case, the closed span, $\overline{span(V)}$, consists of all bounded and continuous complex valued functions which are the limits of uniformly convergent trigonometric series on the real line. Therefore, $\overline{span(V)}$ is the set of all almost periodic functions in sense of Bohr [12], [13]. Another way to construct a new sequence space is by using the concept of two-normed space introduced by Orlicz in [27]. Namely, let U be any normed sequence space endowed with the norm $\|\cdot\|$, and let $\|\cdot\|^*$ be another norm on U. Then $(U, \|\cdot\|, \|\cdot\|^*)$ is said to be a two-normed space, where we assume that the norm $\|\cdot\|$ is coarse from the norm $\|\cdot\|^*$. Also, $\|\cdot\|$ and $\|\cdot\|^*$ are called the basic and starred norm on U, respectively. Clearly, if we take $\|\cdot\| = \|\cdot\|^*$, then $(U, \|\cdot\|, \|\cdot\|^*)$ reduces to the normed space $(U, \|\cdot\|)$. More information about

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two-normed spaces can be found in [27]. Other standard techniques are the construction of quotient spaces and cartesian products (for more, see [1], [2]-[6], [11], [22], [24], [25], [26], [29], [32], [34]).

Let U be a normed space which has not been obtained from any space using standard techniques. If we can derive a new space from U via standard techniques, then U is called base space. For example, the space V mentioned above and the spaces ℓ_∞, c, c_0, ℓ_p, f and f_0 which are called bounded, convergent, null, absolutely p summable, almost convergent and almost null convergent sequence spaces of complex numbers, respectively, are base spaces.

A matrix $A \in (\ell_\infty : c)$ is called a Schur matrix [28]. We recall the following important properties of Schur matrices.

Proposition 1.1. *If A is a Schur matrix, then $\lim_n a_{mn} = \alpha_n$ exists for each n and if $x \in \ell_\infty$, then $\lim_m (Ax)_m = \sum_n \alpha_n x_n$.*

Proposition 1.2. *$A \in (\ell_\infty : c_0)$ if and only if $\lim_m \sum_n |a_{mn}| = 0$.*

The main purpose of the present paper is to construct a new class of base spaces called *rf*-, *rf*$_0$-, *zrf*- and *zrf*$_0$- convergent sequence spaces and to analyze the duals and some classes of matrix mappings on these spaces. Furthermore, we introduce the $B_{\widehat{R}} - core$ of a complex valued sequence and examine some inclusion theorems related to this new type of core.

The rest of paper is organized as follows. In Section 2, we summarize the basic knowledge regarding almost convergence in the literature. In Section 3, we show that the spaces *rf* and *zrf* are isometrically isomorphic, and investigate some algebraic and topological properties of the spaces *rf*, *rf*$_0$, *zrf* and *zrf*$_0$. In Section 4, we state and prove theorems determining the duals of the spaces *rf*, *rf*$_0$, *zrf* and *zrf*$_0$. Then, we study the classes $(rf : \ell_\infty)$, $(rf : c)$, $(\ell_\infty : rf)$ and $(c : rf)$. In Section 5, we characterize the matrix mappings from *zrf* into any given sequence space by means of dual summability methods. We also determine the classes $(zrf : \ell_\infty)$, $(zrf : c)$, $(\ell_\infty : zrf)$ and $(c : zrf)$. In the final section, we introduce the $B_{\widehat{R}} - core$ of a complex valued sequence.

Now, we provide some notations and definitions in order to explain our idea. For simplicity, through all the text, we shall write $\sum_n$, $\sup_n$, $\limsup_n$ and $\lim_n$ instead of $\sum_{n=0}^{\infty}$, $\sup_{n\in\mathbb{N}}$, $\limsup_{n\to\infty}$ and $\lim_{n\to\infty}$ where $\mathbb{N} = \{0, 1, 2, ...\}$. By w we denote the space of all complex valued sequences. Each vector subspace of w is called a sequence space. Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we can say that A defines a matrix mapping from λ to μ, and we denote it by writing $A \in (\lambda : \mu)$, if for every sequence $x = (x_k)$ in λ, the sequence $Ax = \{(Ax)_n\}$ (the A- transform of x), is in μ, where k runs from 0 to ∞. The domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}, \tag{1.1}$$

which is a sequence space. If we take $\lambda = c$, then c_A is called the convergence domain of A. We write the limit of Ax as $A - \lim_n x_n = \lim_n \sum_{k=0}^{\infty} a_{nk} x_k$, and A is called regular if $\lim Ax = \lim x$ for every $x \in c$. $A = (a_{nk})$ is called a triangle matrix if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. If A is a triangle matrix, then one can easily see that the sequence spaces λ_A and λ are linearly isomorphic, i.e., $\lambda_A \cong \lambda$. A sequence space λ with a linear topology is called a K- space provided that each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. If λ is a complete linear metric space then it is called an FK-space. Any FK-space whose topology is normable is called a BK- space [10].

We now recall some well-known triangle and regular matrices.

The Cesàro matrix of order one $C = (c_{nk})$ is a lower triangular matrix defined by

$$c_{nk} = \begin{cases} \frac{1}{n+1}, & 0 \le k \le n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. A matrix U is called a *generalized Cesàro matrix* if it is obtained from C by shifting rows. Let $\theta : \mathbb{N} \to \mathbb{N}$. Then $U = (u_{nk})$ is defined by

$$u_{nk} = \begin{cases} \frac{1}{n+1}, & \theta(n) \le k \le \theta(n) + n, \\ 0, & \text{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}$.

Let us suppose that G is the set of all such matrices obtained by using all possible functions θ. The following lemma given by Butkovic, Kraljevic and Sarapa [15] characterizes the set of almost convergent sequences.

Lemma 1.3. *The set f of all almost convergent sequences is equal to the set $\cap_{U\in G} c_U$.*

One of the best known regular matrices is $R = (r_{nk})$, the Riesz matrix, which is defined by

$$r_{nk} = \begin{cases} \frac{r_k}{R_n}, & 0 \le k \le n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, where (r_k) is real sequence with $r_0 > 0$, $r_k \ge 0$ and $R_n = \sum_{k=0}^{n} r_k$. The Riesz matrix R is regular if and only if $R_n \to \infty$ as $n \to \infty$ [28].

The matrix Z^p defined by

$$Z^p = (z^p_{nk}) = \begin{cases} p, & n = k, \\ 1-p, & n-1 = k, \\ 0, & \text{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}$ and $p \in \mathbb{R} - \{-1\}$ is called a Zweier matrix [14].

For $i = 1, 2, ...,$ let $A^i = (a^i_{nk})$ be an infinite matrix of complex numbers. Let $\mathscr{A}$ denote the sequence of matrices (A^i). For a sequence $x = (x_k)$, the double sequence $t = (t^i_n)$ defined by $t^i_n = \sum_{k=1}^{\infty} a^i_{nk} x_k$ is called the $\mathscr{A}$- transform of $x = (x_k)$ whenever the series converges for all n and i. A sequence $x = (x_k)$ is said to be $\mathscr{A}$- summable to some number l, if $t = (t^i_n)$ converges to l as n tends to ∞, uniformly for $i = 1, 2, \ldots$. Furthermore, the number l is said to be the $\mathscr{A}$- limit of $x = (x_k)$, written $\mathscr{A} - \lim x_k = l$.

2. The sequence space *f* of almost convergent sequences

In this section, we deal with the sequence space f of almost convergent sequences. First of all, we recall the definition of the Banach limit $L : \ell_\infty \to \mathbb{R}$ that is a continuous linear functional on ℓ_∞ such that the following statements hold for any sequences $x = (x_k)$ and $y = (y_k)$ [8]:

(i) $L(ax_k + by_k) = aL(x_k) + bL(y_k)$, $a, b \in \mathbb{R}$;

(ii) if $x_k \ge 0$ for all $k \in \mathbb{N}$, then $L(x_k) \ge 0$;

(iii) $L(Sx) = L(x)$, where S is the shift operator defined by $(Sx)_k = x_{k+1}$;

(iv) $L(e) = 1$, where $e = (1, 1, ...)$.

A bounded sequence x is called almost convergent to $a \in \mathbb{R}$ if all Banach limits of the sequence x are equal to $a \in \mathbb{C}$, and this is denoted by $f - \lim x_k = a$ [23]. Given a sequence $x = (x_k)$, we define $t_{mn}(x)$ for all $m, n \in \mathbb{N}$ by $t_{mn}(x) = \frac{1}{m+1}\sum_{i=0}^{m}(S^i x)_n$. Lorentz [23] proved that $f - \lim x_k = a$ if and only if $\lim_m t_{mn}(x) = a$, uniformly in n. By f and f_0, we denote the space of all almost convergent and almost null sequences, respectively, i.e.,

$$f = \left\{ x = (x_k) \in \ell_\infty : \exists a = \lim_m \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} \in \mathbb{C}, \text{ uniformly in } n \right\}$$

and

$$f_0 = \left\{ x = (x_k) \in \ell_\infty : \lim_m \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = 0, \text{ uniformly in } n \right\}.$$

In [23], Lorentz obtained the necessary and sufficient conditions for an infinite matrix to contain f in its convergence domain. These are standard Silverman - Toeplitz conditions for regularity, with the additional following condition:

$$\lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{n,k+1}| = 0. \tag{2.1}$$

3. A generalization of the definition of almost convergence by means of the sequence spaces *rf* and *rf*$_0$

Almost convergence can be defined as the intersection of convergence field of a Cesàro matrix that is obtained by displacement of the lines of the first-order Cesàro matrix. Let $v \in \mathbb{N}$ and $x = (x_k) \in \ell_\infty$. Then, let us define the matrix $S^v = (s^v_{nk})$ as follows:

$$s^v_{nk} = \begin{cases} 1, & n + v = k, \\ 0, & \text{otherwise.} \end{cases}$$

The sequence $(S^v x) = (S^0 x, S^1 x, S^2 x, ..., S^v x, ...)$ is called the shifted transforms sequence of x, obtained by S. Thus, almost convergence has the same meaning as the convergence of first-order Cesàro average of the shifted transform sequence $(S^v x) = (S^0 x, S^1 x, S^2 x, ..., S^v x, ...)$ to a fixed sequence for each v. We will denote

$$f_T = \left\{ x \in \ell_\infty : \lim_k [T(S^v x)]_k = a \in \mathbb{C}, v = 0, 1, 2, ... \right\}$$

the set of all T- convergent sequences.

In particular, if we take $a^i_{nk} = \frac{r_k}{R_n}$ if $i \leq k \leq n + i$ and 0 otherwise, then the sequence x is said to rf-summable to a if $(R^i x)_n = \frac{1}{R_n} \sum_{k=0}^{n} r_k x_{k+i}$ converges to a as $n \to \infty$, uniformly for $i = 1, 2, ...$. By rf and rf_0, we denote the sequence spaces of all rf- convergent and null rf- convergent sequences, respectively, i.e.,

$$rf = \left\{ x = (x_k) \in \ell_\infty : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} = a, \text{ uniformly in } n \right\}, \tag{3.1}$$

$$rf_0 = \left\{ x = (x_k) \in \ell_\infty : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} = 0, \text{ uniformly in } n \right\}. \tag{3.2}$$

The spaces rf_0 and rf are not obtained by the convergence field of an infinite matrix. By taking this into consideration, we can say that these are base spaces. In addition to rf_0 and rf, we define two new types of convergent sequence spaces, zrf and zrf_0, as the sets of all sequences such that their Z^p- transforms are in rf and rf_0, respectively, that is,

$$zrf = \left\{ x = (x_k) \in w : \lim_m \sum_{k=0}^{m} \frac{r_k}{R_m} [p x_{k+n} + (1 - p) x_{k+n-1}] = a, \text{ uniformly in } n \right\}$$

and

$$zrf_0 = \left\{ x = (x_k) \in w : \lim_m \sum_{k=0}^{m} \frac{r_k}{R_m} [p x_{k+n} + (1 - p) x_{k+n-1}] = 0, \text{ uniformly in } n \right\}.$$

Clearly, the sets zrf_0 and zrf are not base spaces. Now, let us define the sequence $y = (y_k)$, which will be frequently used, as the Z^p - transform of a sequence $x = (x_k)$, i.e.,

$$y_k = p x_k + (1 - p) x_{k-1} \text{ for all } k \in \mathbb{N}, p \in \mathbb{R} - \{-1\}. \tag{3.3}$$

We should emphasize here that the sequence spaces rf and rf_0 can be reduced to the classical almost convergent sequence spaces of real numbers f and f_0 respectively, in the case $r_k = 1$ for all $k \in \mathbb{N}$. Thus, the properties and results related to the sequence spaces *rf* and rf_0 are more general than the corresponding implications for the spaces f and f_0 respectively.

Lorentz [23] proved that if the regular matrix method A has the property (2.1) then f and *rf* are equivalent. But, if $\lim_n R_n$ is not equal to ∞, then the Riesz matrix R is not a Toeplitz matrix. Therefore, in general, the spaces *rf* and rf_0 are different from f and f_0.

Lemma 3.1. *The sets rf and rf_0 are Banach spaces with the norm*

$$\|x\|_{rf} = \|x\|_{rf_0} = \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} \right|, \text{ uniformly in } n. \tag{3.4}$$

Proof. Clearly, the norm conditions are satisfied. We consider only the space *rf*, since the fact that rf_0 is a Banach space can be proved in a similar way. Let us suppose that the sequence (x_k^i) is Cauchy in the space *rf*. Then there exists $n_0 \in \mathbb{N}$ such that for all $i, j \geq n_0$ we have

$$|t_{mn}^i(x) - t_{mn}^j(x)| < \epsilon, \tag{3.5}$$

where $t_{mn}^i(x) = \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n}^i(x)$ and $t_{mn}^j(x) = \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n}^j(x)$. This shows that for every $m, n \in \mathbb{N}$ the sequence $(t_{mn}^i(x))$ is Cauchy in $\mathbb{R}$. Let $\lim_i t_{mn}^i(x) = t_{mn}(x)$. By (3.5), $|t_{mn}(x) - t_{mn}^j(x)| < \epsilon$, hence $t_{mn}^i(x)$ converges to $t_{mn}(x)$.

It is easy to see that $t_{mn}(x) \in rf$. This completes the proof. □

Theorem 3.2. *The sets zrf and zrf_0 are linear spaces with the co-ordinatewise addition and scalar multiplication, and BK- spaces with the norm defined by*

$$\|x\|_{zrf_0} = \|x\|_{zrf} = \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k [p x_{k+n} + (1-p) x_{k+n-1}] \right|, \text{ uniformly in } n. \tag{3.6}$$

Proof. The first part of the theorem is clear. We will only prove the second part. Since (3.3) holds and *rf*, rf_0 are Banach spaces (see Lemma 3.1) and the matrix Z^p is normal, the conclusion follows by Theorem 4.3.3 of Wilansky [33]. □

Theorem 3.3. *The sequence spaces rf and rf_0 are isometrically isomorphic to the spaces zrf and zrf_0, respectively.*

Proof. We consider only the spaces *rf* and *zrf*, since the discussion regarding rf_0 and zrf_0 is similar. In order to prove the fact that $rf \cong zrf$, we should show the existence of a linear bijection between these spaces. Consider the transformation T defined, with the notation of (3.3), from *zrf* to *rf* by $x \mapsto y = Tx$. The linearity of T is clear. Furthermore, it is trivial that x is equal to $\theta = (0, 0, ...)$ whenever $Tx = \theta$ and hence T is injective.

Let $y = (y_k) \in rf$, $\mathfrak{B}^k = \sum_{j=0}^{k} (-1)^{k-j} \frac{(1-p)^{k-j}}{p^{k-j+1}}$, $\mathfrak{B}^{k-1} = \sum_{j=0}^{k-1} (-1)^{k-j} \frac{(1-p)^{k-j}}{p^{k-j+1}}$. If we define the sequence $x = (x_k)$ by $(\mathfrak{B}^k y_j)$ then we see that T is surjective. Since

$$\begin{aligned} \|x\|_{zrf} &= \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k [p x_{k+n} + (1-p) x_{k+n-1}] \right| \\ &= \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k [p \mathfrak{B}^k y_j + (1-p) \mathfrak{B}^{k-1} y_j] \right| \\ &= \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k y_k \right| = \|y\|_{rf}, \end{aligned}$$

it follows that T is norm preserving, so the spaces *zrf* and *rf* are isometrically isomorphic. □

We recall that a sequence space λ is said to be solid if and only if $\ell_\infty\lambda \subset \lambda$ [14].

Theorem 3.4. *The space rf is not a solid sequence space.*

Proof. If we take $u = (u_k) = (1, 1, ...)$ and $v = (v_k) = (1, 0, 1, 1, 0, 0, ...)$ for $k \in \mathbb{N}$ then we see that $u \in rf$, $v \in \ell_\infty$ and $rf - \lim v = \lim_m \frac{1}{R_m}\sum_{k=0}^{m} r_k x_{k+n} = \infty$. It means that $uv = v \notin rf$, that is, rf is not solid. □

Theorem 3.5. *The inclusions $c_0 \subset rf_0 \subset rf \subset \ell_\infty$, $rf_0 \subset zrf_0$ and $rf \subset zrf$ hold for $(r_k) = (1)$.*

Proof. The proof of the theorem is clear so we omit it. □

It is known that a set $\lambda \subset w$ is said to be convex if for all $x, y \in \lambda$, $M = \{z \in w : z = tx + (1-t)y, 0 \leq t \leq 1\} \subset \lambda$.

Theorem 3.6. *The sets rf, rf_0, zrf and zrf_0 are convex spaces.*

Proof. The proof of the theorem is clear from the definition of convexity. □

4. Duals

In this section, by using techniques in [7], we state and prove theorems determining the α-, β- and γ-duals of the spaces rf_0, rf, zrf_0 and zrf.

For the sequence spaces λ and μ, define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x = (x_k) \in \lambda\}. \tag{4.1}$$

If we take $\mu = \ell_1$ then the set $S(\lambda, \ell_1)$ is called the α- dual of λ; similarly, the sets $S(\lambda, cs)$, $S(\lambda, bs)$ are called the β- and γ- dual of λ and are denoted by λ^α, λ^β and λ^γ, respectively.

Theorem 4.1 ([20])**.** *If $\lambda \subset \mu$, then $\mu^\xi \subset \lambda^\xi$ where $\xi \in \{\alpha, \beta, \gamma\}$.*

As a consequence of Theorems 3.5 and 4.1, we obtain that the $\xi \in \{\alpha, \beta, \gamma\}$- duals of the spaces rf and rf_0 is the space ℓ_1.

We state the following results which will be used in the computation of the β- dual of the sets zrf and zrf_0.

Lemma 4.2. *Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (rf : \ell_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{4.2}$$

Proof. Suppose that $\sup_n \sum_k |a_{nk}| < \infty$ and $x \in rf \subset \ell_\infty$. Then Ax exists because of the fact that $(a_{nk})_{k\in\mathbb{N}} \in rf^\beta = \ell_1$ for every $k \in \mathbb{N}$. Therefore $\|(Ax)_n\|_{\ell_\infty} = \sup_n |\sum_k a_{nk}x_k| \leq \sup_n \sum_k |a_{nk}|\|x\|_{rf} < \infty$. The converse is proved similarly, so we omit the details. □

Proposition 4.3. *Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (rf : c)$ if and only if*

$$\lim_n \sum_k a_{nk} = a, \tag{4.3}$$

$$\lim_n a_{nk} = a_k \quad (k \in \mathbb{N}), \tag{4.4}$$

and

$$\lim_n \sum_k |\triangle(a_{nk} - a_k)| = 0. \tag{4.5}$$

Lemma 4.4. *Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (\ell_\infty : rf)$ if and only if* (4.2) *and*

$$rf - \lim_n a_{nk} = a_k, \forall k \in \mathbb{N}, \tag{4.6}$$

$$\sum_k \left| \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k} - a_k \right| = 0, \textit{ uniformly in } n \tag{4.7}$$

hold.

Proof. Suppose that $A \in (\ell_\infty : rf)$. Then the necessity of condition (4.2) is obtained similarly as in the case of Lemma 4.2. Now, let the equations $e_n^{(k)} = \delta_{nk}, (n \in \mathbb{N})$, and $a_k = rf - \lim Ae^{(k)}$ hold. Then $(Ae^{(k)})_n = a_{nk}$ implies that a_k is equal to $rf - \lim a_{nk}$. Suppose that $(B^{(n)}) = (b_{mk}^{(n)}), b_{mk}^{(n)} = \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k}$. The matrix $B^{(n)}$ satisfies the conditions of the Schur theorem. Since $A \in (\ell_\infty : rf)$ for all $x \in \ell_\infty$, the sequence $(B^n x)_m = \sum_k \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k} x_k = \frac{1}{R_m} \sum_{i=0}^{m} \sum_k r_i a_{n+i,k} x_k = (S_m Ax)_n$ converges for $m \to \infty$, uniformly in n. Therefore, $\lim_m b_{mk}^{(n)} = a_k$ for each k, n, whence $\lim_m (S_m Ax)_n = \lim_m (B^{(n)} x)_m = \sum_k a_k x_k$, uniformly in n. It follows that $rf - \lim Ax = \sum_k a_k x_k$ holds for each x.

Now, define the sequence $(C^{(n)})$ by $C_{mk}^{(n)} = \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k} - a_k$. It is clear that $(C^{(n)} x)_m = (S_m Ax)_n - \sum_k a_k x_k$, therefore $\lim_m (C^{(n)} x)_m = 0$, uniformly in n, for all $x \in \ell_\infty$. Consequently, $\lim_m \sum_k | \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k} - a_k | = 0$, uniformly in n.

Conversely, suppose that the matrix A satisfies the conditions (4.2), (4.6) and (4.7). Then, we have

$$\left| (S_m Ax)_n - \sum_k a_k x_k \right| \leq \|x\| \left(\sum_k \left| \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k} - a_k \right| \right) \tag{4.8}$$

uniformly in n for all $x \in \ell_\infty$. Therefore, $rf - \lim Ax = \sum_{k=0}^{\infty} a_k x_k$. This completes the proof. □

Lemma 4.5. *Let $A = (a_{nk})$ be an infinite matrix. Then $A \in (c : rf)$ if and only if*

$$\sup_m \sum_k \left| \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{ik} \right| < \infty, \quad (k, m \in \mathbb{N}), \tag{4.9}$$

$$\lim_m \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k} = a_k \in \mathbb{C}, \textit{ uniformly in } n \tag{4.10}$$

and

$$\lim_m \frac{1}{R_m} \sum_k \sum_{i=0}^{m} r_i a_{n+i,k} = a, \textit{ uniformly in } n, \tag{4.11}$$

hold.

Proof. Suppose that $A \in (c : rf)$ and $t_{mn}(x) = \frac{1}{R_m} \sum_{i=0}^{m} r_i \sigma_i(x)$, where $\sigma_i(x) = \sum_k a_{n+i,k} x_k$. It is clear that $\sigma_i \in \sigma^* = \{\sigma : \sigma : c \to \mathbb{C} \text{ is linear and continuous}, \forall i, n \in \mathbb{N}\}$. Hence, for $m = 0, 1, ..., t_{mn}(x) \in \sigma^*$. Since $A \in (c : rf)$, we can write $\lim_m t_{mn}(x) = t(x)$ uniformly in n. It follows that $x \in c$ and we have $(t_{mn}(x)) \in \ell_\infty$ for all $k \in \mathbb{N}$. Therefore, the sequence $(\|t_{mn}\|)$ is bounded according to the uniform convergence principle.

Let us define the sequence $y = (y_k)$ as follows:

$$y_k = \begin{cases} sgn \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k}, & 0 \leq k \leq r, \\ 0, & r < k, \end{cases}, \quad \forall k, r \in \mathbb{N}.$$

One can easily see that $\|y\|_c = 1$ and $|t_{mn}(y)| = \frac{1}{R_m} \sum_k | \sum_{i=0}^{m} r_i a_{n+i,k}|$, hence we obtain $|t_{mn}(y)| \leq \|t_{mn}\| \|y\| = \|t_{mn}\|$. This shows that $\frac{1}{R_m} \sum_k | \sum_{i=0}^{m} r_i a_{n+i,k}| \leq \|t_{mn}\|$, that is, (4.9) holds.

Consider the sequences $e = (1)$ and $(e_k) = (0, ..., 0, 1, 0, ...)$, where 1 is in the k^{th} position of the sequence e_k. Since $e, (e_k) \in c$ it is easy to see that $\lim_m t_{mn}(e)$ and $\lim_m t_{mn}(e_k)$ are convergent uniformly in n. We conclude that (4.10) and (4.11) hold.

Conversely, suppose that (4.9), (4.10) and (4.11) are satisfied. Let x be in c. Then the following inequality holds:

$$|t_{mn}(x)| \leq \frac{1}{R_m} \sum_k \left| \sum_{i=0}^{m} r_i a_{n+i,k} \right| \|x\|.$$

Now,

$$t_{mn}(x) = \frac{1}{R_m} \sum_{i=0}^{m} \sum_k r_i a_{n+i,k} x_i = \frac{1}{R_m} \sum_k \sum_{i=0}^{m} r_i a_{n+i,k} x_i.$$

From (4.9) we can write $|t_{mn}(x)| \leq K\|x\|$, where $K \in \mathbb{R}$. Moreover, by considering the function $t_{mn}(x) \in \sigma^*$ for $m = 1, 2, ...$, we can see that the sequence $(\|t_{mn}\|)$ is bounded. By (4.10) and (4.11), the limits $\lim_m t_{mn}(e)$ and $\lim_m t_{mn}(e_k)$ exist. Since the set $\{e, e_0, e_1, ...\}$ is fundamental in c, $\lim_m t_{mn}(x) = t_n(x)$. Furthermore, $t_n(x)$ is linear and continuous from c to $\mathbb{C}$.

The expression $t_n(x)$ can be written as

$$t_n(x) = b \left[t_n(e) - \sum_k t_n(e_k) \right] + \sum_k x_k t_n(e_k), \tag{4.12}$$

where $b = \lim x_k$ ([21]). The equalities $t_n(e) = a$ and $t_n(e_k) = a_k$ hold for $k = 0, 1, ...,$ from (4.10) and (4.11), respectively. Thus, for $k = 0, 1, ...,$ and every $x \in c$ we can write

$$\lim_m t_{mn}(x) = t(x) \text{ and } t(x) = b \left[a - \sum_k a_k \right] + \sum_k a_k x_k.$$

Furthermore, since $t_{mn} \in \sigma^*$, we obtain

$$t_{mn}(x) = b \left[t_{mn}(e) - \sum_k t_{mn}(e_k) \right] + \sum_k x_k t_{mn}(e_k). \tag{4.13}$$

From (4.12) and (4.13) we can easily see that $(t_m(x)) \to t(x)$, uniformly in n since $\lim_m t_{mn}(e) = a$ and $\lim_m t_{mn}(e_k) = a_k$. This completes the proof. □

Lemma 4.6 ([7]). *Let $D = (d_{nk})$ be defined via a sequence $a = (a_k) \in w$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $U = (u_{nk})$, by*

$$d_{nk} = \begin{cases} \sum_{j=k}^{n} a_j v_{jk}, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then,

$$\{\lambda_U\}^\gamma = \{a = (a_k) \in w : D \in (\lambda : \ell_\infty)\}$$

and

$$\{\lambda_U\}^\beta = \{a = (a_k) \in w : D \in (\lambda : c)\}.$$

Let us define the sets d_i, $i = 1, 2, 3$ as follows:

$$d_1 = \left\{ (u_k) \in w : \sup_n \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{(-1)^{j-k}(1-p)^{j-k}}{p^{j-k+1}} u_j \right| < \infty \right\},$$

$$d_2 = \left\{ (u_k) \in w : \ \exists \lim_n \sum_{j=k}^{n} \frac{(-1)^{j-k}(1-p)^{j-k}}{p^{j-k+1}} u_j \right\},$$

$$d_3 = \left\{ (u_k) \in w : \lim_n \sum_{j=k}^{n} \left| \Delta \left(\frac{(-1)^{j-k}(1-p)^{j-k}}{p^{j-k+1}} u_j - u_k \right) \right| = 0 \right\}.$$

Theorem 4.7. *The β- dual of the spaces zrf and zrf_0 is the set $\mathscr{D} = \bigcap_{i=1}^{3} d_i$.*

Proof. Define the matrix $V = (v_{nk})$ via the sequence $u = (u_k) \in w$ by

$$v_{nk} = \begin{cases} \sum_{j=k}^{n} (-1)^{j-k} \frac{(1-p)^{j-k}}{p^{j-k+1}} u_j, & 0 \le k \le n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Given that $x_k = \mathfrak{B}^k y_j$, we find that

$$\sum_{k=0}^{n} u_k x_k = \sum_{k=0}^{n} r_i \sum_{j=k}^{n} (-1)^{j-k} \frac{(1-p)^{j-k}}{p^{j-k+1}} u_j y_k = (Vy)_n, \quad n \in \mathbb{N}. \tag{4.14}$$

From (4.14), we see that $ux = (u_k x_k) \in cs$ whenever $x = (x_k) \in zrf$ if and only if $Vy \in c$ whenever $y = (y_k) \in rf$. Then we derive by Proposition 4.3 that $zrf^{\beta} = zrf_0^{\beta} = \mathscr{D}$. □

5. Some matrix mappings related to the space *zrf*

In this section, we characterize the matrix mappings from *zrf* into any given sequence space via a new concept of dual summability methods.

Suppose that the sequences $u = (u_k)$ and $v = (v_k)$ are connected via (3.3) and let $z = (z_k)$ be the A-transform of the sequence $u = (u_k)$ and $t = (t_k)$ be the B-transform of the sequence $v = (v_k)$ i.e.,

$$z_k = (Au)_k = \sum_k a_{nk} u_k, \quad (k \in \mathbb{N}) \tag{5.1}$$

and

$$t_k = (Bv)_k = \sum_k b_{nk} v_k, \quad (k \in \mathbb{N}). \tag{5.2}$$

It is clear here that B is applied to the Z^p- transform of the sequence $u = (u_k)$, while A is directly applied to the terms of the sequence $u = (u_k)$. Then it is easy to see that methods A and B are essentially different (see [9]).

Let us assume that the matrix product BZ^p exists (this is a much weaker assumption than that of matrix B belonging to any matrix class, in general). If z_k becomes t_k (or t_k becomes z_k), under the application of the formal summation by parts, then the methods A and B as in (5.1) and (5.2) are called Zweier dual type matrices. This leads us to the fact that BZ^p exists and is equal to A and $(BZ^p)u = B(Z^p u)$. This statement is equivalent to the relation

$$b_{nk} = \sum_{j=k}^{n} (-1)^{j-k} \frac{(1-p)^{j-k}}{p^{j-k+1}} a_{nj} \text{ or } a_{nk} = p b_{nk} + (1-p) b_{n,k+1} \tag{5.3}$$

for all $n, k \in \mathbb{N}$.

Now, we give the following theorem concerning Zweier dual matrices:

Theorem 5.1. *Let $A = (a_{nk})$ and $B = (b_{nk})$ be dual matrices of new type and μ be any given sequence space. Let $\lim_m b_{nm} = 0$ for all $n \in \mathbb{N}$ and $(a_{nk})_{k\in\mathbb{N}} \in \ell_1$. Then $A \in (zrf : \mu)$ if and only if $B \in (rf : \mu)$.*

Proof. Suppose that $A = (a_{nk})$ and $B = (b_{nk})$ are Zweier dual matrices, that is to say (5.3) holds, and μ is any given sequence space. Additionally, note that the spaces *zrf* and *rf* are isomorphic.

Let $A \in (zrf : \mu)$ and take any $y = (y_k) \in rf$. Then BZ^p exists and $(a_{nk})_{k\in\mathbb{N}} \in \mathscr{D}$, which implies that $(b_{nk})_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, By exists for each $y \in rf$. Using the hypothesis and letting $m \to \infty$ in the equality

$$\sum_{k=0}^{m} b_{nk} y_k = \sum_{k=0}^{m-1} (pb_{nk} + (1-p)b_{n,k+1})x_k + pb_{nm}x_m, \ \forall m, n \in \mathbb{N}, \tag{5.4}$$

we obtain $By = Ax$. It follows that $B \in (rf : \mu)$.

Conversely, suppose that (5.4) and $B \in (rf : \mu)$ hold for every fixed $k \in \mathbb{N}$ and take any $x = (x_k) \in zrf$. Then, Ax exists. Therefore, from

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m}\sum_{j=k}^{m} (-1)^{j-k}\frac{(1-p)^{j-k}}{p^{j-k+1}} a_{nj} y_k = \sum_{k=0}^{m} b_{nk} y_k \quad (n \in \mathbb{N}), \tag{5.5}$$

by taking $m \to \infty$ we obtain that $Ax = By$. From here, it is clear that $A \in (zrf : \mu)$. □

Theorem 5.2. *Suppose that the elements of the infinite matrices $D = (d_{nk})$ and $E = (e_{nk})$ are connected via the relation*

$$e_{nk} = pd_{nk} + (1-p)d_{n-1,k}, \quad (n, k \in \mathbb{N}) \tag{5.6}$$

and let μ be any given sequence space. Then $D \in (\mu : zrf)$ if and if only $E \in (\mu : rf)$.

Proof. Suppose that $x = (x_k) \in \mu$. Since (5.6) holds and

$$\frac{1}{R_n}\sum_{k=0}^{n} r_k[pd_{n,k+i}x_{k+i} + (1-p)d_{n-1,k+i}x_{k+i}] = \frac{1}{R_n}\sum_{k=0}^{n} r_k(e_{n,k+i}x_{k+i}),$$

we obtain for $n \to \infty$ that $\|Dx\|_{zrf} = \|Ex\|_{rf}$. □

The following propositions are consequences of Proposition 4.3, Lemma 4.4 and Theorems 5.1 and 5.2:

Proposition 5.3. *Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. Then $A = (a_{nk}) \in (zrf : \ell_\infty)$ if and only if $(a_{nk})_{k\in\mathbb{N}} \in zrf^\beta$ for all $n \in \mathbb{N}$ and*

$$\sup_n \sum_k \left| \sum_{j=k}^{n} (-1)^{j-k}\frac{(1-p)^{j-k}}{p^{j-k+1}} a_{nj} \right| < \infty. \tag{5.7}$$

Proposition 5.4. *Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. Then $A = (a_{nk}) \in (zrf : c)$ if and only if $(a_{nk})_{k\in\mathbb{N}} \in zrf^\beta$ for all $n \in \mathbb{N}$, (5.7) and following statements hold:*

(i) $\lim_n \sum_k \sum_{j=k}^{n} (-1)^{j-k}\frac{(1-p)^{j-k}}{p^{j-k+1}} a_{nj} = a$,

(ii) $\lim_n \sum_{j=k}^{n} (-1)^{j-k}\frac{(1-p)^{j-k}}{p^{j-k+1}} a_{nj} = a_k$ *for each fixed* $k \in \mathbb{N}$,

(iii) $\lim_n \sum_k \left| \triangle \left(\sum_{j=k}^{n} (-1)^{j-k}\frac{(1-p)^{j-k}}{p^{j-k+1}} a_{nj} - a_k \right) \right| = 0$.

Proposition 5.5. *Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. Then $A = (a_{nk}) \in (\ell_\infty : zrf)$ if and only if following statements hold:*

(i) $\sup_n \sum_k |pa_{nk} + (1-p)a_{n-1,k}| < \infty$,

(ii) $rf - \lim_n pa_{nk} + (1-p)a_{n-1,k} = a_k$ *exists for each fixed* $k \in \mathbb{N}$,

(iii) $\lim_n \sum_k |\frac{1}{R_n} \sum_{i=0}^{n} r_i a_{\nu+i,k} - a_k| = 0$, *uniformly in* ν.

Proposition 5.6. *Let* $A = (a_{nk})$ *be an infinite matrix of real or complex numbers. Then* $A = (a_{nk}) \in (c : zrf)$ *if and only if*

(i) $\sup_n \sum_k |pa_{nk} + (1-p)a_{n-1,k}| < \infty$,

(ii) $\lim_q \frac{1}{R_q} \sum_{i=0}^{q} r_i (pa_{n+i,k} + (1-p)a_{n+i-1,k}) = a_k$ *exists, uniformly in* n,

(iii) $\lim_q \frac{1}{R_q} \sum_{i=0}^{q} r_i \sum_n (pa_{k+i,n} + (1-p)a_{k+i-1,n}) = a$ *exists, uniformly in* k.

6. Core theorems of new type

In this section, we give some core theorems related to the rf - and zrf - cores.

Let $x = (x_k)$ be a sequence in $\mathbb{C}$ and $\mathfrak{R}_k$ be the least convex closed region of the complex plane containing $x_k, x_{k+1}, x_{k+2}, \ldots$. The Knopp Core (or $\mathcal{K} - core$) of x is defined by the intersection of all $\mathfrak{R}_k, (k = 1, 2, ...)$ (see [17]). In [30], it is shown that

$$\mathcal{K} - core(x) = \bigcap_{z \in \mathbb{C}} B_x(z)$$

for any bounded sequence x, where $B_x(z) = \{w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z|\}$.

Let E be a subset of $\mathbb{N}$. The natural density δ of E is defined by

$$\delta(E) = \lim_n \frac{1}{n} |\{k \leq n : k \in E\}|,$$

where $|\{k \leq n : k \in E\}|$ denotes the number of elements of E not exceeding n. A sequence $x = (x_k)$ is said to be statistically convergent to a number ℓ if $\delta(\{k : |x_k - \ell| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$. In this case, we write $st - \lim x = \ell$ [31]. By st and st_0 we denote the space of all statistically convergent and statistically null sequences, respectively.

In [19], Fridy and Orhan introduced the notion of the statistical core(or $st - core$) of a complex valued sequence and showed that, if x is a statistically bounded sequence x, then

$$st - core(x) = \bigcap_{z \in \mathbb{C}} C_x(z),$$

where $C_x(z) = \{w \in \mathbb{C} : |w - z| \leq st - \limsup_k |x_k - z|\}$.

In this section, we will consider complex valued sequences, and by $\ell_\infty(\mathbb{C})$ we denote the space of all such sequences which are bounded.

Following Knopp, a core theorem characterizes a class of matrices for which the core of the transformed sequence is included in the core of original sequence. For example, the Knopp Core Theorem [17] states that $\mathcal{K} - core(Ax) \subset \mathcal{K} - core(x)$ for all real valued sequences x whenever A is a positive matrix in the class $(c, c)_{reg}$.

Now, we introduce the $B_{\widehat{R}} - core$ of a complex valued sequence and characterize the class of matrices such that $B_{\widehat{R}} - core(Ax) \subseteq \mathcal{K} - core(x)$, $\mathcal{K} - core(Ax) \subseteq B_{\widehat{R}} - core(x)$, $B_{\widehat{R}} - core(Ax) \subseteq B_{\widehat{R}} - core(x)$ and $B_{\widehat{R}} - core(Ax) \subseteq st - core(x)$ for all $x \in \ell_\infty(\mathbb{C})$.

Considering

$$t_{mn}(x) = \frac{1}{R_m} \sum_{i=0}^{m} r_i x_{i+n},$$

we can define the $B_{\widehat{R}} - core$ of a complex sequence as follows.

Definition 6.1. Let H_n be the least closed convex hull containing $t_{mn}(x), t_{m+1,n}(x), t_{m+2,n}(x), \ldots$. Then, the $B_{\widehat{R}} - core$ of x is the intersection of all H_n, i.e.,

$$B_{\widehat{R}} - core(x) = \bigcap_{n=1}^{\infty} H_n.$$

Note that we have defined the $B_{\widehat{R}} - core$ of x by the $\mathcal{K} - core$ of the sequence $(t_{mn}(x))$, Consequently, we can obtain the following theorem which is analogue of that for the $\mathcal{K} - core$ in [30]:

Theorem 6.2. *For any $z \in \mathbb{C}$, let*

$$G_x(z) = \left\{ w \in \mathbb{C} : |w - z| \leq \limsup_{m} \sup_{n} |t_{mn}(x) - z| \right\}.$$

Then, for any $x \in \ell_\infty$,

$$B_{\widehat{R}} - core(x) = \bigcap_{z \in \mathbb{C}} G_x(z).$$

Now, we need to characterize the classes $A \in (c : zrf)_{reg}$ and $(st \bigcap \ell_\infty : zrf)_{reg}$. For brevity, through all the text we write $\tilde{a}(m,n,k) = \tilde{a}$ instead of

$$\frac{1}{R_m} \sum_{i=0}^{m} r_i a_{i+n,k}$$

for all $m, n, k \in \mathbb{N}$.

Lemma 6.3. $A \in (c : zrf)_{reg}$ *if and only if* (4.9) *and* (4.10) *of the* Lemma 4.5 *hold with $a_k = 0$ for all $k \in \mathbb{N}$ and*

$$\lim_m \sum_k \tilde{a} = 1, \text{ uniformly in } n. \tag{6.1}$$

Lemma 6.4. $A \in (st \bigcap \ell_\infty : zrf)_{reg}$ *if and only if* $A \in (c : zrf)_{reg}$ *and*

$$\lim_m \sum_{k \in E} |\tilde{a}| = 0, \text{ uniformly in } n, \tag{6.2}$$

for every $E \subset \mathbb{N}$ with natural density zero.

Proof. Let $A \in (st \cap \ell_\infty : zrf)_{reg}$. Then $A \in (c : zrf)_{reg}$ immediately follows from the fact that $c \subset st \cap \ell_\infty$. Now, define the sequence $t = (t_k)$ for $x \in \ell_\infty$ by

$$t_k = \begin{cases} x_k, & k \in E, \\ 0, & k \notin E, \end{cases}$$

where E is any subset of $\mathbb{N}$ with $\delta(E) = 0$. Then $st - \lim t_k = 0$ and $t \in st_0$, so we have $At \in zrf_0$. On the other hand, since $(At)_n = \sum_{k \in E} a_{nk} t_k$, the matrix $B = (b_{nk})$ defined by

$$b_{nk} = \begin{cases} a_{nk}, & k \in E, \\ 0, & k \notin E \end{cases}$$

for all n, must belong to the class (ℓ_∞, zrf_0). Hence, the necessity of (6.2) is clear.

Conversely, let $x \in st \cap \ell_\infty$ and $st - \lim x = \ell$. Then, for any given $\varepsilon > 0$, the set $E = \{k : |x_k - \ell| \geq \varepsilon\}$ has density zero and $|x_k - \ell| \leq \varepsilon$ if $k \notin E$. From here, it is clear that

$$\sum_k \tilde{a} x_k = \sum_k \tilde{a}(x_k - \ell) + \ell \sum_k \tilde{a}. \tag{6.3}$$

Since

$$\left|\sum_k \tilde{a}(x_k-\ell)\right| \leq \|x\| \sum_{k\in E} |\tilde{a}| + \varepsilon\|A\|,$$

by letting $m\to\infty$ in (6.3) and using (6.1) with (6.2), we have

$$\lim_m \sum_k \tilde{a}x_k = \ell.$$

This implies that $A \in (st \bigcap \ell_\infty : zrf)_{reg}$ and the proof is completed. □

Lemma 6.5 ([18], Corollary 12)**.** *Let $\mathcal{A} = \{a_{mk}(n)\}$ defined by $a_{mk}(n) = \tilde{a}$ for all $m, n, k \in \mathbb{N}$ be a matrix satisfying $\|\mathcal{A}\| = \|a_{mk}(n)\| < \infty$ and $\limsup_m \sup_n |a_{mk}(n)| = 0$. Then there exists $y \in \ell_\infty$ with $\|y\| \leq 1$ such that*

$$\limsup_m \sup_n \sum_k \tilde{a}y_k = \limsup_m \sup_n \sum_k |\tilde{a}|.$$

Theorem 6.6. *$B_{\widehat{R}} - core(Ax) \subseteq \mathcal{K} - core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (c : zrf)_{reg}$ and*

$$\limsup_m \sup_n \sum_k |\tilde{a}| = 1. \tag{6.4}$$

Proof. Suppose that $B_{\widehat{R}} - core(Ax) \subseteq \mathcal{K} - core(x)$ and take $x \in c$ with $\lim x = \ell$. Then, since $\mathcal{K} - core(x) \subseteq \{\ell\}$, $B_{\widehat{R}} - core(Ax) \subseteq \{\ell\}$, $zrf - \lim Ax = \ell$, which means that $A \in (c : zrf)_{reg}$. It follows that the matrix $\mathcal{A} = \tilde{a}$ satisfies the conditions of Lemma 6.5. Thus, there exists $y \in \ell_\infty$ with $\|y\| \leq 1$ such that

$$\left\{w \in \mathbb{C} : |w| \leq \limsup_m \sup_n \sum_k \tilde{a}y_k\right\} = \left\{w \in \mathbb{C} : |w| \leq \limsup_m \sup_n \sum_k |\tilde{a}|\right\}.$$

On the other hand, since $\mathcal{K} - core(y) \subseteq A_1^*(0)$, by the hypothesis we have

$$\left\{w \in \mathbb{C} : |w| \leq \limsup_m \sup_n \sum_k |\tilde{a}|\right\} \subseteq A_1^*(0) = \{w \in \mathbb{C} : |w| \leq 1\},$$

which implies (6.4).

Conversely, let $w \in B_{\widehat{R}} - core(Ax)$. Then, for any given $z \in \mathbb{C}$, we can write

$$\begin{aligned}
|w-z| &\leq \limsup_m \sup_n |t_{mn}(Ax) - z| = \limsup_m \sup_n \left|z - \sum_k \tilde{a}x_k\right| \\
&\leq \limsup_m \sup_n \left|\sum_k \tilde{a}(z-x_k)\right| + \limsup_m \sup_n |z| \left|1 - \sum_k \tilde{a}\right| \\
&= \limsup_m \sup_n \left|\sum_k \tilde{a}(z-x_k)\right|.
\end{aligned}$$

Now, let $L(x) = \limsup_k |x_k - z|$. Then, for any $\varepsilon > 0$, $|x_k - z| \leq L(x) + \varepsilon$ whenever $k \geq k_0$. Hence, one can write

$$\begin{aligned}
\left|\sum_k \tilde{a}(z-x_k)\right| &= \left|\sum_{k<k_0} \tilde{a}(z-x_k) + \sum_{k\geq k_0} \tilde{a}(z-x_k)\right| \\
&\leq \sup_k |z-x_k| \sum_{k<k_0} |\tilde{a}| + [L(x)+\varepsilon] \sum_{k\geq k_0} |\tilde{a}| \\
&\leq \sup_k |z-x_k| \sum_{k<k_0} |\tilde{a}| + [L(x)+\varepsilon] \sum_{k\geq k_0} |\tilde{a}|.
\end{aligned}$$

Therefore, applying $\limsup_m \sup_n$, in light of the hypothesis we get

$$|w-z| \leq \limsup_m \sup_n \big| \sum_k \tilde{a}(z-x_k) \big| \leq L(x)+\varepsilon,$$

which means that $w \in \mathcal{K}-core(x)$. □

Theorem 6.7. *$B_{\widehat{R}}-core(Ax) \subseteq st-core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (st \bigcap \ell_\infty : zrf)_{reg}$ and (6.4) holds.*

Proof. First, we suppose that $B_{\widehat{R}}-core(Ax) \subseteq st-core(x)$ for all $x \in \ell_\infty$. By taking $x \in st \bigcap \ell_\infty$, one can see that $A \in (st \bigcap \ell_\infty : zrf)_{reg}$. Also, since $st-core(x) \subseteq \mathcal{K}-core(x)$ for any x ([16]), the necessity of the condition (6.4) follows from Theorem 6.6.

Conversely, suppose $A \in (st \bigcap \ell_\infty : zrf)_{reg}$ and (6.4) holds, and take $w \in B_{\widehat{R}}-core(Ax)$. Now, let $\beta = st-\limsup |z-x_k|$. If we set $E=\{k : |x_k - z| \geq \beta+\varepsilon\}$, then $\delta(E)=0$ and $|z-x_k| \leq \beta+\varepsilon$ whenever $k \notin E$. From here, we obtain

$$\begin{aligned}\left|\sum_k \tilde{a}(z-x_k)\right| &= \left|\sum_{k\in E} \tilde{a}(z-x_k) + \sum_{k\notin E} \tilde{a}(z-x_k)\right| \\ &\leq |z-x_k| \sum_{k\in E} |\tilde{a}| + \sum_{k\notin E} |\tilde{a}||z-x_k| \\ &\leq |z-x_k| \sum_{k\in E} |\tilde{a}| + [\beta+\varepsilon] \sum_{k\notin E} |\tilde{a}|.\end{aligned}$$

By applying the operator $\limsup_m \sup_n$ and using the hypothesis with (6.2) and (6.4), we find that

$$\limsup_m \sup_n \left|\sum_k \tilde{a}(z-x_k)\right| \leq \beta+\varepsilon. \tag{6.5}$$

Thus, (6.5) implies that $|w-z| \leq \beta+\varepsilon$. Since ε is arbitrary, this means that $w \in st-core(x)$, which completes the proof. □

Author contributions

Sections 1, 2 and 3 represent the joint work of Zarife Zararsız and Mehmet Şengönül. The last section is the joint work of Kuddusi Kayaduman and Zarife Zararsız. All authors read and approved the final manuscript.

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