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On the Determinants and Inverses of $R-$circulant Matrices with the Biperiodic Fibonacci and Lucas Numbers

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Abstract. In this paper, we present a new generalization to compute determinants and inverses of $r-$circulant matrices $\mathcal{Q}_n = circ_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} q_1, \left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} q_2, \ldots, \left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)$ and $\mathcal{L}_n = circ_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} l_1, \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} l_2, \ldots, \left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)$ whose entries are the biperiodic Fibonacci and the biperiodic Lucas numbers, respectively. Also, we express determinants of the matrices $\mathcal{Q}_n$ and $\mathcal{L}_n$ by using only the biperiodic Fibonacci and the biperiodic Lucas numbers.

1. Introduction and Preliminaries

For $n \in \mathbb{N}_0$, the Fibonacci and Lucas numbers are defined by $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$ with the initial conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$, respectively. During the recent years, the researchers have studied the generalizations, representations and applications of the Fibonacci and Lucas numbers [2–7]. For example, Edson and Yayenie introduced a new generalization of Fibonacci sequence [4], $\{q_n\}_{n\in\mathbb{N}_0}$

$$q_0 = 0, \quad q_1 = 1, \quad q_{n+2} = \begin{cases} aq_{n+1} + q_n, & \text{if } n \text{ is even} \\ bq_{n+1} + q_n, & \text{if } n \text{ is odd} \end{cases}, \tag{1}$$

where a, b are nonzero real numbers and $n \in \mathbb{N}_0$. They also developed an extended Binet formula which is

$$q_n = \left(\frac{a^{1-\xi(n)}}{ab^{\lfloor \frac{n}{2} \rfloor}}\right) \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2}$$

where $n \in \mathbb{N}_0$ and $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$. Then, Bilgici [5] defined a new generalization of the Lucas numbers, $\{l_n\}_{n\in\mathbb{N}_0}$, as

$$l_0 = 2, \quad l_1 = a, \quad l_{n+2} = \begin{cases} bl_{n+1} + l_n, & \text{if } n \text{ is even} \\ al_{n+1} + l_n, & \text{if } n \text{ is odd} \end{cases}, \tag{3}$$

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where a, b are nonzero real numbers and $n \in \mathbb{N}_0$. Also he developed the Binet formula of this sequence as

$$l_n = \left(\frac{a^{\xi(n)}}{ab^{\lfloor \frac{n+1}{2} \rfloor}}\right)(\alpha^n + \beta^n), \quad n \in \mathbb{N}_0. \tag{4}$$

In equation (2) and (4), $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$ and $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$ are the roots of the characteristic equation of $x^2 - abx - ab = 0$. Throughout this paper, we assume that a and b are positive real numbers.

Now we give some preliminaries related our study. A matrix $C = \left[c_{ij}\right] \in M_{n,n}(\mathbb{C})$ is called circulant matrix if it is of the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i, \\ c_{n+j-i}, & j < i \end{cases}.$$

The determinant and inverse of a nonsingular circulant matrix $A = circ(a_0, a_1, \dots, a_{n-1})$ can be given as [17]

$$\det A = \prod_{r=0}^{n-1} g(w^r), \quad A^{-1} = circ(b_0, b_1, \dots, b_{n-1}), \tag{5}$$

where $b_s = \frac{1}{n}\sum_{r=0}^{n-1} g(w^r)^{-1} w^{-rs}$, where $g(x) = \sum_{i=0}^{n-1} a_i x^i$, $w = \exp(\frac{2\pi i}{n})$ and $s = 0, 1, \dots, n-1$. In recent years, there have been several studies on the norms, determinants and inverses of circulant and $r-$circulant matrices [9–13, 15, 16, 18, 19]. For instance, Köme and Yazlik studied the spectral norms of r-circulant matrices with the biperiodic Fibonacci and Lucas numbers [14]. Yazlik and Taskara presented the norms of an $r-$circulant matrix with the generalized $k-$Horadam numbers [15]. Shen et al. gave the determinants and inverses of $r-$circulant matrices with the Fibonacci and Lucas numbers [16].

2. Determinant and inverse of $r-$circulant matrix with the biperiodic Fibonacci numbers

Definition 2.1. *An $(n \times n)$ $r-$circulant matrix with biperiodic Fibonacci numbers entries is defined by*

$$Q_n = \begin{bmatrix} \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} q_1 & \left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} q_2 & \left(\frac{b}{a}\right)^{\frac{\xi(4)}{2}} q_3 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n \\ r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n & \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} q_1 & \left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} q_2 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n-1} \\ r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n-1} & r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n & \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} q_1 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r\left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} q_2 & r\left(\frac{b}{a}\right)^{\frac{\xi(4)}{2}} q_3 & r\left(\frac{b}{a}\right)^{\frac{\xi(5)}{2}} q_4 & \cdots & \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} q_1 \end{bmatrix} \tag{6}$$

The following theorem gives us the values of the determinant of this matrix can be expressed by using only the biperiodic Fibonacci numbers.

Theorem 2.2. *Let $n \geq 3$. Assume that $Q_n = circ_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} q_1, \left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} q_2, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)$ is $r-$circulant. Then,*

$$\begin{aligned} \det Q_n &= \left(q_1^2 - r\sqrt{\frac{b}{a}} q_2 \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n\right)\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-2} + r\sum_{k=1}^{n-2}\left[\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}} q_{n-k+1} - \sqrt{\frac{b}{a}} q_2 \left(\frac{b}{a}\right)^{\frac{\xi(n-k-1)}{2}} q_{n-k}\right)\right. \\ &\quad \left. \times \left(r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n - \sqrt{\frac{b}{a}} q_0\right)^k \left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-k-2}\right], \end{aligned} \tag{7}$$

where $r \neq \frac{q_1}{\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}}$.

Proof. It is clear that $\det \mathcal{Q}_3 = q_1^3 + r\left(\sqrt{\frac{b}{a}}q_2\right)^3 + r^2q_3^3 - 3r\sqrt{\frac{b}{a}}q_1q_2q_3$ and it satisfies the equation (7). For $n > 3$, we define the matrices

$$\mathcal{P}_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -r\sqrt{\frac{b}{a}}\frac{q_2}{q_1} & 0 & 0 & 0 & \dots & 0 & 1 \\ -r & 0 & 0 & 0 & \dots & 1 & -\sqrt{ab} \\ 0 & 0 & 0 & 0 & \dots & -\sqrt{ab} & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & -\sqrt{ab} & \dots & 0 & 0 \\ 0 & 1 & -\sqrt{ab} & -1 & \dots & 0 & 0 \end{pmatrix} \tag{8}$$

and

$$\mathcal{R}_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^{n-2} & 0 & 0 & \dots & 0 & 0 \\ 0 & \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^{n-3} & 0 & 0 & \dots & 0 & 1 \\ 0 & \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^{n-4} & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right) & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{9}$$

Using these matrices, we get

$$\begin{aligned} \mathcal{S}_n &= \mathcal{P}_n\mathcal{Q}_n\mathcal{R}_n \\ &= \begin{pmatrix} q_1 & g'_n & \beta_n & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_5 & \beta_4 & \beta_3 \\ 0 & g_n & q_1 - \frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n}{q_1} & \gamma_n & \gamma_{n-1} & \cdots & \gamma_6 & \gamma_5 & \gamma_4 \\ & & \sigma_n & \theta_n & & & & & \\ & & & \sigma_n & \theta_n & & & & \\ & & & & \sigma_n & \ddots & & & \\ & & & & & \ddots & \theta_n & & \\ & & & & & & \sigma_n & \theta_n & \\ & & & & & & & \sigma_n & \theta_n \\ & & & & & & & & \sigma_n \end{pmatrix}, \end{aligned}$$

where

$$g'_n = \sum_{k=2}^{n}\left(\frac{b}{a}\right)^{\frac{\xi(k-1)}{2}} q_k \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^{n-k},$$

$$g_n = q_1 - \frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n}{q_1} + r\sum_{k=1}^{n-2}\left(\left(\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}}q_{n-k+1} - \frac{\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}}q_{n-k}}{q_1}\right)\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^k\right),$$

$$\gamma_m = r\left(\frac{b}{a}\right)^{\frac{\xi(m-1)}{2}}q_m - \frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(m-2)}{2}}q_{m-1}}{q_1}, \quad \text{for } m = 4, 5, \ldots, n,$$

$$\beta_s = \left(\frac{b}{a}\right)^{\frac{\xi(s-1)}{2}}q_s, \quad \text{for } s = 3, 4, \ldots, n,$$

$$\sigma_n = \sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n \text{ and } \theta_n = q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}.$$

It can be seen from (8) and (9) that

$$\det\mathcal{P}_n = (-1)^{\frac{(n-1)(n-2)}{2}}$$

and

$$\det\mathcal{R}_n = \begin{cases} \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^{n-2}, & n-1 \equiv 1, 2 (\bmod 4) \\ -\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^{n-2}, & n-1 \equiv 0, 3 (\bmod 4). \end{cases}$$

Hence, for all $n > 3$, we obtain

$$\det\mathcal{P}_n \det\mathcal{R}_n = (-1)^n\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^{n-2}.$$

Since $\mathcal{S}_n$ is an upper triangular matrix, we have

$$\det\mathcal{S}_n = q_1 g_n\left[\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)\right]^{n-2} = (-1)^{n-2}q_1 g_n\left[\left(r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0\right)\right]^{n-2}.$$

By using the identity

$$\det\mathcal{S}_n = \det\mathcal{P}_n \det\mathcal{Q}_n \det\mathcal{R}_n,$$

we get

$$(-1)^{n-2}q_1 g_n\left[\left(r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0\right)\right]^{n-2} = (-1)^n\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^{n-2}\det\mathcal{Q}_n.$$

If $q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1} \neq 0$, we obtain

$$\begin{aligned}
\det Q_n &= q_1 g_n \left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-2} \\
&= \left(q_1^2 - r\sqrt{\frac{b}{a}} q_2 \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n\right)\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-2} + r\sum_{k=1}^{n-2}\left[\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}} q_{n-k+1} - \sqrt{\frac{b}{a}} q_2 \left(\frac{b}{a}\right)^{\frac{\xi(n-k-1)}{2}} q_{n-k}\right)\right. \\
&\quad \left. \times \left(r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n - \sqrt{\frac{b}{a}} q_0\right)^k \left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-k-2}\right].
\end{aligned}$$

□

Lemma 2.3. *Let $\mathcal{A} = (a_{ij})$ be the $(n-2)\times(n-2)$ matrix defined by*

$$a_{ij} = \begin{cases} q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}, & j = i+1 \\ \sqrt{\frac{b}{a}} q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n, & i = j \\ 0, & \text{otherwise} \end{cases}$$

such that $r \neq \frac{\sqrt{\frac{b}{a}} q_0}{\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n}$. Then, inverse of matrix $\mathcal{A}$, $\mathcal{A}^{-1} = \left(a'_{ij}\right)$, can be given by

$$a'_{ij} = \begin{cases} \dfrac{\left(-\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)\right)^{j-i}}{\left(\sqrt{\frac{b}{a}} q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n\right)^{j-i+1}}, & j \geq i \\ 0, & \text{otherwise} \end{cases}.$$

Theorem 2.4. *Let $Q_n = circ_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} q_1, \left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} q_2, \ldots, \left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)$ be $r-$circulant ($n \geq 3$ and $0 \neq r \in \mathbb{C}$) such that $r \neq \frac{\sqrt{\frac{b}{a}} q_0}{\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n}, \frac{q_1}{\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}}$. Then*

$$Q_n^{-1} = circ_r\left(\omega_1, \omega_2, \ldots, \omega_n\right),$$

where

$$\omega_1 = \frac{1}{g_n} - v\left(\frac{\sqrt{ab}}{\sqrt{\frac{b}{a}} q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n} - \frac{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}}{\left(\sqrt{\frac{b}{a}} q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^2}\right) + \frac{r\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n - \sqrt{\frac{b}{a}} q_2 \left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n-1}\right)}{q_1 g_n\left(\sqrt{\frac{b}{a}} q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)},$$

$$\omega_2 = -\frac{v}{\sqrt{\frac{b}{a}} q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n} - \frac{\sqrt{\frac{b}{a}} q_2}{q_1 g_n},$$

$$\omega_3 = (-1)^{n-1}\frac{v\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-3}}{r\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^{n-2}} + \frac{1}{q_1 g_n}\sum_{k=1}^{n-3}(-1)^{n-k}\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}} q_{n-k+1} - \sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}} q_{n-k}\right)$$

$$\times\frac{\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-k-3}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^{n-k-2}},$$

$$\omega_4 =(-1)^{n}\frac{v\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-4}\left(\sqrt{\frac{b}{a}}q_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_{n+2}\right)}{r\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^{n-2}} + \frac{\left(q_1\sqrt{\frac{b}{a}}q_4 - \sqrt{\frac{b}{a}}q_2 q_3\right)\sqrt{ab}}{q_1 g_n\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)}$$

$$+\frac{\left(\sqrt{\frac{b}{a}}q_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_{n+2}\right)}{q_1 g_n}\sum_{k=2}^{n-3}\left[(-1)^{k-1}\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(k+4)}{2}} q_{k+3} - \sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(k+3)}{2}} q_{k+2}\right)\right.$$

$$\left.\times\frac{\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{k-2}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^{k}}\right],$$

$$\omega_j =(-1)^{n-j}\frac{v}{r}\left(\frac{\left(\sqrt{\frac{b}{a}}q_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_{n+2}\right)\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-j}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^{n-j+2}} - \frac{\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-j+2}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^{n-j+3}}\right)$$

$$+\frac{1}{q_1 g_n}\left[\sum_{k=1}^{n-j}(-1)^{n-j+k+1}\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}} q_{n-k+1} - \sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}} q_{n-k}\right)\right.$$

$$\times\left(\frac{\left(\sqrt{\frac{b}{a}}q_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_{n+2}\right)\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-j-k}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^{n-j-k+2}} - \frac{\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^{n-j-k+2}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^{n-j-k+3}}\right)$$

$$+\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(j+1)}{2}} q_j - \sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(j)}{2}} q_{j-1}\right)\left(\frac{\sqrt{\frac{b}{a}}q_0\sqrt{ab} - q_1 + r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n-1}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^{2}}\right)$$

$$\left.+\frac{q_1\left(\frac{b}{a}\right)^{\frac{\xi(j)}{2}} q_{j-1} - \sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(j-1)}{2}} q_{j-2}}{\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n}\right],\quad \text{for } j = 5, 6, \ldots, n-1,$$

$$\omega_n = \frac{v}{r}\left(\frac{\sqrt{\frac{b}{a}}q_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_{n+2}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^2} - \frac{\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}\right)^2}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^3}\right) + \frac{1}{q_1 g_n}\left[\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n - \sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n-1}\right)\right.$$
$$\left.\times\left(\frac{\sqrt{\frac{b}{a}}q_0\sqrt{ab} - q_1 + r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n-1}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n\right)^2}\right) + \frac{q_1\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n-1} - \sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_{n-2}}{\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n}\right],$$

and

$$g_n = q_1 - \frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n}{q_1} + r\sum_{k=1}^{n-2}\left(\left(\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}} q_{n-k+1} - \frac{\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}} q_{n-k}}{q_1}\right)\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}}\right)^k\right),$$

$$v = \frac{q_1 g_n - q_1^2 + r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n}{q_1 g_n}.$$

Proof. Let

$$\mathcal{U}_n = \begin{pmatrix} 1 & -\frac{g_n'}{q_1} & u_{13} & u_{14} & u_{15} & \cdots & u_{1,n-1} & u_{1n} \\ 0 & 1 & -\frac{q_1}{g_n} + \frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n}{q_1 g_n} & \frac{\zeta_n}{q_1 g_n} & \frac{\zeta_{n-1}}{q_1 g_n} & \cdots & \frac{\zeta_5}{q_1 g_n} & \frac{\zeta_4}{q_1 g_n} \\ & 0 & 1 & & & & & \\ & & 0 & 1 & & & & \\ & & & 0 & 1 & & & 0 \\ & & & & & \ddots & & \\ 0 & & & & & \ddots & 1 & \\ & & & & & & 0 & 1 \end{pmatrix},$$

where

$$\zeta_m = -rq_1\left(\frac{b}{a}\right)^{\frac{\xi(m+1)}{2}} q_m + r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(m)}{2}} q_{m-1}, \quad \text{for } m = 4, 5, \ldots, n,$$

$$u_{13} = -\frac{\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n}{q_1} + \frac{g_n'}{q_1 g_n}\left(q_1 - \frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n}{q_1}\right),$$

$$u_{1j} = -\frac{\left(\frac{b}{a}\right)^{\frac{\xi(n-j+2)}{2}} q_{n-j+3}}{q_1} + \frac{g_n'}{q_1 g_n}\left(r\left(\frac{b}{a}\right)^{\frac{\xi(n-j+3)}{2}} q_{n-j+4} - \frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-j+2)}{2}} q_{n-j+3}}{q_1}\right), \quad \text{for } j = 4, 5, \ldots, n,$$

$$g_n' = \sum_{k=2}^{n}\left(\frac{b}{a}\right)^{\frac{\xi(k+1)}{2}} q_k\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} q_{n+1}}\right)^{n-k},$$

and

$$g_n = q_1 - \frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n}{q_1} + r\sum_{k=1}^{n-2}\left(\left(\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}}q_{n-k+1} - \frac{\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}}q_{n-k}}{q_1}\right)\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n - \sqrt{\frac{b}{a}}q_0}{q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^k\right).$$

Let $\mathcal{H} = \mathrm{diag}(q_1, g_n)$. Then we get

$$\mathcal{P}_n\mathcal{Q}_n\mathcal{R}_n\mathcal{U}_n = \mathcal{H} \oplus \mathcal{A},$$

where the matrix $\mathcal{A}$ is given in the Lemma 2.3 and $\oplus$ denotes the direct sum. Let $\mathcal{T}_n = \mathcal{R}_n\mathcal{U}_n$. Then we have

$$\mathcal{Q}_n^{-1} = \mathcal{T}_n\left(\mathcal{H}^{-1} \oplus \mathcal{A}^{-1}\right)\mathcal{P}_n.$$

Let

$$\mathcal{Q}_n^{-1} = circ_r(\omega_1, \omega_2, \ldots, \omega_n).$$

Since the last row of $\mathcal{T}_n$ is

$$\left(0, 1, 1 - \frac{q_1}{g_n} + \frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n}{q_1 g_n}, \frac{\zeta_n}{q_1 g_n}, \frac{\zeta_{n-1}}{q_1 g_n}, \ldots, \frac{\zeta_5}{q_1 g_n}, \frac{\zeta_4}{q_1 g_n}\right),$$

using Lemma 2.3, we can find the entries of the last row of $\mathcal{Q}_n^{-1}$ as

$$r\omega_2 = -\frac{rv}{\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n} - \frac{r\sqrt{\frac{b}{a}}q_2}{q_1 g_n},$$

$$r\omega_3 = (-1)^{n-1}\frac{v\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}\right)^{n-3}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^{n-2}} + \frac{r}{q_1 g_n}\sum_{k=1}^{n-3}(-1)^{n-k}\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}}q_{n-k+1} - \sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}}q_{n-k}\right)$$
$$\times\frac{\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}\right)^{n-k-3}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^{n-k-2}}$$

$$r\omega_4 = (-1)^n\frac{v\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}\right)^{n-4}\left(\sqrt{\frac{b}{a}}q_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_{n+2}\right)}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^{n-2}} + \frac{r\left(q_1\sqrt{\frac{b}{a}}q_4 - \sqrt{\frac{b}{a}}q_2q_3\right)\sqrt{ab}}{q_1 g_n\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)}$$
$$+\frac{r\left(\sqrt{\frac{b}{a}}q_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_{n+2}\right)}{q_1 g_n}\sum_{k=2}^{n-3}\left[(-1)^{k-1}\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(k+4)}{2}}q_{k+3} - \sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(k+3)}{2}}q_{k+2}\right)\right.$$
$$\left.\times\frac{\left(q_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}\right)^{k-2}}{\left(\sqrt{\frac{b}{a}}q_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^{k}}\right],$$

$$
\begin{aligned}
r\omega_j =&(-1)^{n-j}v\left(\frac{\left(\sqrt{\frac{b}{a}}q_2-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_{n+2}\right)\left(q_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}\right)^{n-j}}{\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^{n-j+2}}-\frac{\left(q_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}\right)^{n-j+2}}{\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^{n-j+3}}\right)\\
&+\frac{r}{q_1g_n}\left[\sum_{k=1}^{n-j}(-1)^{n-j+k+1}\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}}q_{n-k+1}-\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}}q_{n-k}\right)\right.\\
&\times\left(\frac{\left(\sqrt{\frac{b}{a}}q_2-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_{n+2}\right)\left(q_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}\right)^{n-j-k}}{\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^{n-j-k+2}}-\frac{\left(q_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}\right)^{n-j-k+2}}{\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^{n-j-k+3}}\right)\\
&+\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(j+1)}{2}}q_j-\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(j)}{2}}q_{j-1}\right)\left(\frac{\sqrt{\frac{b}{a}}q_0\sqrt{ab}-q_1+r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n-1}}{\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^2}\right)\\
&\left.+\frac{q_1\left(\frac{b}{a}\right)^{\frac{\xi(j)}{2}}q_{j-1}-\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(j-1)}{2}}q_{j-2}}{\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n}\right],\text{for } j=5,6,\dots,n-1,
\end{aligned}
$$

$$
\begin{aligned}
r\omega_n =&v\left(\frac{\sqrt{\frac{b}{a}}q_2-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_{n+2}}{\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^2}-\frac{\left(q_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}\right)^2}{\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^3}\right)+\frac{r}{q_1g_n}\left[\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n-\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n-1}\right)\right.\\
&\left.\times\left(\frac{\sqrt{\frac{b}{a}}q_0\sqrt{ab}-q_1+r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n-1}}{\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^2}\right)+\frac{q_1\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n-1}-\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_{n-2}}{\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n}\right],
\end{aligned}
$$

$$
\omega_1=\frac{1}{g_n}-v\left(\frac{\sqrt{ab}}{\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n}-\frac{q_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}{\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)^2}\right)+\frac{r\left(q_1\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n-\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n-1}\right)}{q_1g_n\left(\sqrt{\frac{b}{a}}q_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n\right)},
$$

where

$$
g_n = q_1-\frac{r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n}{q_1}+r\sum_{k=1}^{n-2}\left(\left(\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}}q_{n-k+1}-\frac{\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}}q_{n-k}}{q_1}\right)\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}q_n-\sqrt{\frac{b}{a}}q_0}{q_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}q_{n+1}}\right)^k\right)
$$

and

$$
v=\frac{q_1g_n-q_1^2+r\sqrt{\frac{b}{a}}q_2\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}q_n}{q_1g_n}.
$$

Since Q^{-1} is r−circulant, the proof is completed. □

3. Determinant and inverse of $r-$circulant matrix with the biperiodic Lucas numbers

Definition 3.1. *An $(n \times n)$ $r-$circulant matrix with biperiodic Lucas numbers entries is defined by*

$$\mathcal{L}_n = \begin{bmatrix} \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} l_1 & \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} l_2 & \left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} l_3 & \dots & \left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n \\ r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n & \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} l_1 & \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} l_2 & \dots & \left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} l_{n-1} \\ r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} l_{n-1} & r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n & \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} l_1 & \dots & \left(\frac{b}{a}\right)^{\frac{\xi(n-2)}{2}} l_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r\left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} l_2 & r\left(\frac{b}{a}\right)^{\frac{\xi(3)}{2}} l_3 & r\left(\frac{b}{a}\right)^{\frac{\xi(4)}{2}} l_4 & \dots & \left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} l_1 \end{bmatrix} \tag{10}$$

The following theorem gives us the values of the determinant of this matrix and shows that they can be expressed by using only the biperiodic Lucas numbers.

Theorem 3.2. *Let $n \geq 3$. Assume that $\mathcal{L}_n = circ_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}} l_1, \left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}} l_2, \dots, \left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)$ is $r-$circulant. Then,*

$$\begin{aligned} \det \mathcal{L}_n &= \left(\left(\frac{b}{a}\right)l_1^2 - rl_2\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{n-2} + r\sum_{k=1}^{n-2}\left[\left(\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}} l_{n-k+1} - l_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}} l_{n-k}\right)\right. \\ &\quad \left. \times\left(r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n - l_0\right)^k\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{n-k-2}\right], \end{aligned} \tag{11}$$

where $r \neq \frac{\sqrt{\frac{b}{a}}l_1}{\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}}$.

Proof. It is clear that $\det \mathcal{L}_3 = \left(\sqrt{\frac{b}{a}}l_1\right)^3 + rl_2^3 + r^2\left(\sqrt{\frac{b}{a}}l_3\right)^3 - 3r\frac{b}{a}l_1l_2l_3$ and it satisfies the equation (11). For $n > 3$, we define the matrices

$$\mathcal{V}_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -r\frac{l_2}{\sqrt{\frac{b}{a}}l_1} & 0 & 0 & 0 & \dots & 0 & 1 \\ -r & 0 & 0 & 0 & \dots & 1 & -\sqrt{ab} \\ 0 & 0 & 0 & 0 & \dots & -\sqrt{ab} & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & -\sqrt{ab} & \dots & 0 & 0 \\ 0 & 1 & -\sqrt{ab} & -1 & \dots & 0 & 0 \end{pmatrix} \tag{12}$$

and

$$\mathcal{G}_n=\begin{pmatrix}1 & 0 & 0 & 0 & \dots & 0 & 0\\ 0 & \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n-l_0}{\sqrt{\frac{b}{a}}l_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}l_{n+1}}\right)^{n-2} & 0 & 0 & \dots & 0 & 0\\ 0 & \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n-l_0}{\sqrt{\frac{b}{a}}l_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}l_{n+1}}\right)^{n-3} & 0 & 0 & \dots & 0 & 1\\ 0 & \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n-l_0}{\sqrt{\frac{b}{a}}l_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}l_{n+1}}\right)^{n-4} & 0 & 0 & \dots & 1 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n-l_0}{\sqrt{\frac{b}{a}}l_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}l_{n+1}}\right) & 0 & 1 & \dots & 0 & 0\\ 0 & 1 & 1 & 0 & \dots & 0 & 0\end{pmatrix}. \tag{13}$$

By using the matrices $\mathcal{V}_n$ and $\mathcal{G}_n$, the proof can be made in a similar way as in the Theorem 2.2. □

Lemma 3.3. *Let $\mathcal{B}=(b_{ij})$ be the $(n-2)\times(n-2)$ matrix defined by*

$$b_{ij}=\begin{cases}\sqrt{\frac{b}{a}}l_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}l_{n+1}, & j=i+1\\ l_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n, & i=j\\ 0, & \textit{otherwise}\end{cases}$$

such that $r\neq\frac{l_0}{\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n}$. Then, inverse of the matrix $\mathcal{B}$, $\mathcal{B}^{-1}=\left(b'_{ij}\right)$, can be given by

$$b'_{ij}=\begin{cases}\frac{\left(-\left(\sqrt{\frac{b}{a}}l_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}l_{n+1}\right)\right)^{j-i}}{\left(l_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n\right)^{j-i+1}}, & j\geq i\\ 0, & \textit{otherwise}\end{cases}.$$

Theorem 3.4. *Let $\mathcal{L}_n=circ_r\left(\left(\frac{b}{a}\right)^{\frac{\xi(1)}{2}}l_1,\left(\frac{b}{a}\right)^{\frac{\xi(2)}{2}}l_2,\dots,\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n\right)$ be $r-$circulant ($n\geq 3$ and $0\neq r\in\mathbb{C}$) such that $r\neq\frac{l_0}{\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n},\frac{\sqrt{\frac{b}{a}}l_1}{\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}l_{n+1}}$. Then*

$$\mathcal{L}_n^{-1}=circ_r\left(\psi_1,\psi_2,\dots,\psi_n\right),$$

where

$$\psi_1=\frac{1}{\varphi_n}-\kappa\left(\frac{\sqrt{ab}}{l_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n}-\frac{\sqrt{\frac{b}{a}}l_1-r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}l_{n+1}}{\left(l_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n\right)^2}\right)+\frac{r\left(\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n-l_2\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}}l_{n-1}\right)}{\sqrt{\frac{b}{a}}l_1\varphi_n\left(l_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n\right)},$$

$$\psi_2=-\frac{\kappa}{l_0-r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n}-\frac{l_2}{\sqrt{\frac{b}{a}}l_1\varphi_n},$$

$$\psi_3 = (-1)^{n-1}\frac{\kappa\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{n-3}}{r\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^{n-2}} + \frac{1}{\sqrt{\frac{b}{a}}l_1\varphi_n}\sum_{k=1}^{n-3}(-1)^{n-k}\left(\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}} l_{n-k+1} - l_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}} l_{n-k}\right)$$
$$\times\frac{\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{n-k-3}}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^{n-k-2}},$$

$$\psi_4 =(-1)^{n}\frac{\kappa\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{n-4}\left(l_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+2)}{2}} l_{n+2}\right)}{r\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^{n-2}} + \frac{\left(\sqrt{\frac{b}{a}}l_1l_4 - l_2\sqrt{\frac{b}{a}}l_3\right)\sqrt{ab}}{\sqrt{\frac{b}{a}}l_1\varphi_n\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)} + \frac{\left(l_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+2)}{2}} l_{n+2}\right)}{\sqrt{\frac{b}{a}}l_1\varphi_n}$$
$$\times\sum_{k=2}^{n-3}\left[(-1)^{k-1}\left(\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(k+3)}{2}} l_{k+3} - l_2\left(\frac{b}{a}\right)^{\frac{\xi(k+2)}{2}} l_{k+2}\right)\frac{\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{k-2}}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^{k}}\right],$$

$$\psi_j =(-1)^{n-j}\frac{\kappa}{r}\left(\frac{\left(l_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+2)}{2}} l_{n+2}\right)\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{n-j}}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^{n-j+2}} - \frac{\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{n-j+2}}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^{n-j+3}}\right)$$
$$+\frac{1}{\sqrt{\frac{b}{a}}l_1\varphi_n}\left[\sum_{k=1}^{n-j}(-1)^{n-j+k+1}\left(\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}} l_{n-k+1} - l_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}} l_{n-k}\right)\right.$$
$$\times\left(\frac{\left(l_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+2)}{2}} l_{n+2}\right)\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{n-j-k}}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^{n-j-k+2}} - \frac{\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^{n-j-k+2}}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^{n-j-k+3}}\right)$$
$$+\left(\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(j)}{2}} l_j - l_2\left(\frac{b}{a}\right)^{\frac{\xi(j-1)}{2}} l_{j-1}\right)\left(\frac{l_0\sqrt{ab} - \sqrt{\frac{b}{a}}l_1 + r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} l_{n-1}}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^2}\right)$$
$$\left.+\frac{\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(j-1)}{2}} l_{j-1} - l_2\left(\frac{b}{a}\right)^{\frac{\xi(j-2)}{2}} l_{j-2}}{l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n}\right], \quad \text{for } j = 5, 6, \ldots, n-1,$$

$$\psi_n = \frac{v}{r}\left(\frac{l_2 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+2)}{2}} l_{n+2}}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^2} - \frac{\left(\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}\right)^2}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^3}\right) + \frac{1}{\sqrt{\frac{b}{a}}l_1 g_n}\left[\left(\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n - l_2\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} l_{n-1}\right)\right.$$
$$\left.\times\left(\frac{l_0\sqrt{ab} - \sqrt{\frac{b}{a}}l_1 + r\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} l_{n-1}}{\left(l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n\right)^2}\right) + \frac{\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(n-1)}{2}} l_{n-1} - l_2\left(\frac{b}{a}\right)^{\frac{\xi(n-2)}{2}} l_{n-2}}{l_0 - r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n}\right],$$

and

$$\varphi_n = \sqrt{\frac{b}{a}}l_1 - \frac{rl_2\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n}{\sqrt{\frac{b}{a}}l_1} + r\sum_{k=1}^{n-2}\left(\left(\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}} l_{n-k+1} - \frac{l_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}} l_{n-k}}{\sqrt{\frac{b}{a}}l_1}\right)\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n - l_0}{\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}}\right)^k\right),$$

$$\kappa = \frac{\sqrt{\frac{b}{a}}l_1\varphi_n - \left(\frac{b}{a}\right)l_1^2 + rl_2\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n}{\sqrt{\frac{b}{a}}l_1\varphi_n}.$$

Proof. Let

$$\mathcal{M}_n = \begin{pmatrix} 1 & -\frac{\varphi_n'}{\sqrt{\frac{b}{a}}l_1} & \mu_{13} & \mu_{14} & \mu_{15} & \cdots & \mu_{1,n-1} & \mu_{1n} \\ 0 & 1 & -\frac{\sqrt{\frac{b}{a}}l_1}{\varphi_n} + \frac{rl_2\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n}{\sqrt{\frac{b}{a}}l_1\varphi_n} & \frac{\lambda_n}{\sqrt{\frac{b}{a}}l_1\varphi_n} & \frac{\lambda_{n-1}}{\sqrt{\frac{b}{a}}l_1\varphi_n} & \cdots & \frac{\lambda_5}{\sqrt{\frac{b}{a}}l_1\varphi_n} & \frac{\lambda_4}{\sqrt{\frac{b}{a}}l_1\varphi_n} \\ & 0 & 1 & & & & & 0 \\ & & 0 & 1 & & & & \\ & & & 0 & 1 & & & \\ & & & & & \ddots & & \\ & & & & & \ddots & 1 & \\ & & & & & & 0 & 1 \end{pmatrix},$$

where

$$\lambda_m = -r\sqrt{\frac{b}{a}}l_1\left(\frac{b}{a}\right)^{\frac{\xi(m)}{2}} l_m + rl_2\left(\frac{b}{a}\right)^{\frac{\xi(m-1)}{2}} l_{m-1}, \quad \text{for } m = 4, 5, \ldots, n,$$

$$\mu_{13} = -\frac{\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n}{\sqrt{\frac{b}{a}}l_1} + \frac{\varphi_n'}{\sqrt{\frac{b}{a}}l_1\varphi_n}\left(\sqrt{\frac{b}{a}}l_1 - \frac{rl_2\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n}{\sqrt{\frac{b}{a}}l_1}\right),$$

$$\mu_{1j} = -\frac{\left(\frac{b}{a}\right)^{\frac{\xi(n-j+3)}{2}} l_{n-j+3}}{\sqrt{\frac{b}{a}}l_1} + \frac{\varphi_n'}{\sqrt{\frac{b}{a}}l_1\varphi_n}\left(r\left(\frac{b}{a}\right)^{\frac{\xi(n-j+4)}{2}} l_{n-j+4} - \frac{rl_2\left(\frac{b}{a}\right)^{\frac{\xi(n-j+3)}{2}} l_{n-j+3}}{\sqrt{\frac{b}{a}}l_1}\right), \text{for } j = 4, \ldots, n,$$

$$\varphi_n' = \sum_{k=2}^{n}\left(\frac{b}{a}\right)^{\frac{\xi(k)}{2}} l_k\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}} l_n - l_0}{\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}} l_{n+1}}\right)^{n-k},$$

and

$$\varphi_n = \sqrt{\frac{b}{a}}l_1 - \frac{rl_2\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n}{\sqrt{\frac{b}{a}}l_1} + r\sum_{k=1}^{n-2}\left(\left(\left(\frac{b}{a}\right)^{\frac{\xi(n-k+1)}{2}}l_{n-k+1} - \frac{l_2\left(\frac{b}{a}\right)^{\frac{\xi(n-k)}{2}}l_{n-k}}{\sqrt{\frac{b}{a}}l_1}\right)\left(\frac{r\left(\frac{b}{a}\right)^{\frac{\xi(n)}{2}}l_n - l_0}{\sqrt{\frac{b}{a}}l_1 - r\left(\frac{b}{a}\right)^{\frac{\xi(n+1)}{2}}l_{n+1}}\right)^k\right).$$

By using the matrix $\mathcal{M}_n$ and given another identities, the proof can be made in a similar way as in the Theorem 2.4. □

4. Conclusion

In recent years, circulant and $r-$circulant matrices have became an attractive topic in mathematics. Several authors studied the generalizations and applications of circulant and $r-$circulant matrices. In this context, our study is also a new generalization of some studies in the literature. For example, if we get $a = b = r = 1$, we obtain the results in [16]. If we get $a = b = k$ and $r = 1$, we obtain the results in [18]. Therefore, this study contributes to the literature by providing essential information for the inverses and determinants of $r-$circulant matrices with the biperiodic Fibonacci and Lucas numbers.

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