Solvability of a system of nonlinear difference equations of higher order*

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Abstract: In this paper, we show that the following higher-order system of nonlinear difference equations,

$$x_n = \frac{x_{n-k}y_{n-k-l}}{y_{n-l}\left(a_n + b_n x_{n-k}y_{n-k-l}\right)}, \ y_n = \frac{y_{n-k}x_{n-k-l}}{x_{n-l}\left(\alpha_n + \beta_n y_{n-k}x_{n-k-l}\right)}, \ n \in \mathbb{N}_0,$$

where $k, l \in \mathbb{N}$, $(a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}, (\alpha_n)_{n \in \mathbb{N}_0}, (\beta_n)_{n \in \mathbb{N}_0}$ and the initial values $x_{-i}, \ y_{-i}, \ i = \overline{1, k+l}$, are real numbers, can be solved and some results in the literature can be extended further. Also, by using these obtained formulas, we investigate the asymptotic behavior of well-defined solutions of the above difference equations system for the case $k = 2, \ l = k$.

Key words: System of nonlinear difference equations, solution of system of difference equations in closed form, asymptotic behavior

1. Introduction and preliminaries

The study of nonlinear difference equations and systems of difference equations has attracted the attention of many authors in recent years (see, e.g., [1–49]). To find the difference equation or system of difference equations that can be solved in closed form is only one of the challenges. Almost all solvable difference equations or their systems have various generalizations of solvable difference equations and systems. That is, when a solvable equation is found, generalizations such as solvability with parameters, solvability with increasing order, solvability with periodic coefficients, and solvability as two-dimensional or three-dimensional systems can be studied. For example, the following difference equations,

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n\left(\pm 1 \pm x_{n-1}x_{n-2}\right)}, \ n \in \mathbb{N}_0, \tag{1.1}$$

were studied by El-Metwally and Elsayed in [10]. Then, in [35], Eq. (1.1) was generalized to the following difference equation:

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1}\left(a_n + b_n x_n x_{n-k}\right)}, \ n \in \mathbb{N}_0, \tag{1.2}$$

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where $k \in \mathbb{N}$. Also, in [32], Eq. (1.2) was extended to the following two-dimensional system of difference equation:

$$x_{n+1} = \frac{x_n y_{n-k}}{y_{n-k+1} (a_n + b_n x_n y_{n-k})}, \ y_{n+1} = \frac{y_n x_{n-k}}{x_{n-k+1} (c_n + d_n x_n y_{n-k})}, \ n \in \mathbb{N}_0, \tag{1.3}$$

where $k \in \mathbb{N}$. Further, system (1.3) is a natural generalization of the difference equation systems given in [12, 15]. Moreover, in [34], the authors studied the next equation:

$$x_n = \frac{x_{n-2} x_{n-k-2}}{x_{n-k} (a_n + b_n x_{n-2} x_{n-k+2})}, \ n \in \mathbb{N}_0, \tag{1.4}$$

where $k \in \mathbb{N}$, which is a natural generalization of the equations given in [10, 11, 14, 20, 21]. On the other hand, in [1], Eq. (1.1) was extended to the next differences systems:

$$x_{n+1} = \frac{x_{n-1} y_{n-2}}{y_n (-1 \pm x_{n-1} y_{n-2})}, \ y_{n+1} = \frac{y_{n-1} x_{n-2}}{x_n (\pm 1 \pm y_{n-1} x_{n-2})}, \ n \in \mathbb{N}_0. \tag{1.5}$$

Some of their solution forms were proved by induction. However, the obtained formulas have not been confirmed by some theoretical explanations.

A natural question is to study both the two-dimensional form of equation (1.4) and more general systems of (1.3) and (1.5) solvable in closed form. Here we study such a system. That is, we deal with the following system of difference equations:

$$x_n = \frac{x_{n-k} y_{n-k-l}}{y_{n-l} (a_n + b_n x_{n-k} y_{n-k-l})}, \ y_n = \frac{y_{n-k} x_{n-k-l}}{x_{n-l} (\alpha_n + \beta_n y_{n-k} x_{n-k-l})}, \ n \in \mathbb{N}_0, \tag{1.6}$$

where $k, l \in \mathbb{N}$, $(a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}, (\alpha_n)_{n \in \mathbb{N}_0}, (\beta_n)_{n \in \mathbb{N}_0}$ and the initial values x_{-i}, y_{-i}, $i = \overline{1, k+l}$, are real numbers. We solve system (1.6) in closed form and determine the asymptotic behavior of solutions for the case $k = 2$, $l = k$. Note that system (1.6) is a natural extension of both Eqs. (1.1), (1.2), and (1.4) and systems (1.3) and (1.5).

This paper is organized as follows. In the following section, we obtain the formulas of the solutions of system (1.6) in closed form and give the forbidden set of the initial values of system (1.6). In Section 3, first we write the formulas of the solutions of system (1.6) when the coefficients of system (1.6) are constant and $k = 2$, $l = k$. Furthermore, in this case, we investigate the asymptotic behavior of the solutions of system (1.6) in detail according to the case of the coefficients a and α.

2. The solutions of the system (1.6)

Let $(x_n, y_n)_{n \geq -k-l}$ be a solution of system (1.6). If at least one of the initial values x_{-i}, y_{-i}, $i = \overline{1, k+l}$, is equal to zero, then the solution of system (1.6) is not defined. For example, if $x_{-k-l} = 0$, then $y_0 = 0$ and so x_l is not defined. Similarly, if $y_{-k-l} = 0$, then $x_0 = 0$ and so y_l is not defined. For $i = \overline{1, k+l-1}$, the other cases are similar. On the other hand, if $x_{n_0} = 0$ $(n_0 \in \mathbb{N}_0)$, $x_n \neq 0$, for $-k - l \leq n \leq n_0 - 1$, and x_m and y_m are defined for $-k - l \leq m \leq n_0 - 1$, then according to the first equation in (1.6) we get that $y_{n_0-k-l} = 0$. If $n_0 - k - l \leq -1$, then $y_{-i_0} = 0$, for $i_0 = \overline{1, k+l}$. If $n_0 > k + l - 1$, then according to the second equation in

(1.6) we have $y_{n_0-2k-l} = 0$. If $n_0 - 2k - l \leq -1$, then $y_{-i_0} = 0$ for $i_0 = \overline{1, k+l}$. Repeating this procedure, we have $y_{-i_0} = 0$ for $i_0 = \overline{1, k+l}$. Similarly, if $y_{n_1} = 0$ ($n_1 \in \mathbb{N}_0$), $y_n \neq 0$, for $-k - l \leq n \leq n_1 - 1$, and x_m and y_m are defined for $-k - l \leq m \leq n_1 - 1$, one can easily show that $x_{-i_1} = 0$ for $i_1 = \overline{1, k+l}$. Thus, for every well-defined solution of system (1.6), we have

$$x_n y_n \neq 0, \; n \geq -k - l, \tag{2.1}$$

if and only if $x_{-i} y_{-i} \neq 0$, for $i = \overline{1, k+l}$. Therefore, by employing the substitution

$$u_n = \frac{1}{x_n y_{n-l}}, \; v_n = \frac{1}{y_n x_{n-l}}, \; n \geq -k, \tag{2.2}$$

we transform system (1.6) into the following nonhomogeneous linear k-order difference equations:

$$u_n = a_n u_{n-k} + b_n, \; v_n = \alpha_n v_{n-k} + \beta_n, \; n \in \mathbb{N}_0. \tag{2.3}$$

If we apply the decomposition of indexes $n \to km + i$, for some $m \geq -1$ and $i = \overline{0, k-1}$, to (2.3), then they become

$$u_{km+i} = a_{km+i} u_{k(m-1)+i} + b_{km+i}, \; v_{km+i} = \alpha_{km+i} v_{k(m-1)+i} + \beta_{km+i}, \; m \in \mathbb{N}_0, \tag{2.4}$$

which are first-order k-equations. The solutions of equations in (2.4) are

$$u_{km+i} = u_{i-k} \prod_{j=0}^{m} a_{kj+i} + \sum_{l=0}^{m} b_{kl+i} \prod_{j=l+1}^{m} a_{kj+i}, \; v_{km+i} = v_{i-k} \prod_{j=0}^{m} \alpha_{kj+i} + \sum_{l=0}^{m} \beta_{kl+i} \prod_{j=l+1}^{m} \alpha_{kj+i}, \; m \geq -1, \tag{2.5}$$

$i \in \{0, 1, \ldots, k-1\}$. If $a_n = a$, $b_n = b$, $\alpha_n = \alpha$, and $\beta_n = \beta$, for every $\in \mathbb{N}_0$, then we have

$$u_{km+i} = \frac{a^{m+1} (u_{i-k} (1-a) - b) + b}{1-a}, \quad v_{km+i} = \frac{\alpha^{m+1} (v_{i-k} (1-\alpha) - \beta) + \beta}{1-\alpha}, \; m \geq -1, \tag{2.6}$$

if $a \neq 1$ and $\alpha \neq 1$, and

$$u_{km+i} = u_{i-k} + b(m+1), \quad v_{km+i} = v_{i-k} + \beta(m+1), \; m \geq -1, \tag{2.7}$$

if $a = 1$ and $\alpha = 1$. From (2.2) it follows that

$$x_n = \frac{1}{u_n y_{n-l}} = \frac{v_{n-l}}{u_n} x_{n-2l}, \quad y_n = \frac{1}{v_n x_{n-l}} = \frac{u_{n-l}}{v_n} y_{n-2l}, \tag{2.8}$$

for $n \geq l - k$, and consequently

$$x_{2lm+i} = x_{i-2l} \prod_{j=0}^{m} \frac{v_{(2j-1)l+i}}{u_{2lj+i}}, \quad y_{2lm+i} = y_{i-2l} \prod_{j=0}^{m} \frac{u_{(2j-1)l+i}}{v_{2lj+i}}, \tag{2.9}$$

for $m \in \mathbb{N}_0$ and $i \in \{l-k, l-k+1, \ldots, 3l-k-1\}$. By the help of the well-known quotient remainder theorem, there exist $k \in \mathbb{N}$ and $s \in \mathbb{N}_0$ such that $n = ks + j_1$ and $j_1 \in \{0, 1, \ldots, k-1\}$. From this and system (2.9), we can write

$$x_{2klm+ks+j_1} = x_{ks+j_1-2kl} \prod_{j=0}^{m} \prod_{n=1}^{k} \frac{v_{2klj+ks+j_1-l(2n-1)}}{u_{2klj+ks+j_1-2l(n-1)}},$$

$$y_{2klm+ks+j_1} = y_{ks+j_1-2kl} \prod_{j=0}^{m} \prod_{n=1}^{k} \frac{u_{2klj+ks+j_1-l(2n-1)}}{v_{2klj+ks+j_1-2l(n-1)}} \qquad (2.10)$$

for $ks + j_1 \in \{2kl - k - l, 2kl - k - l + 1, \ldots, 4kl - k - l - 1\}$, if k is odd, and

$$x_{klm+ks+j_1} = x_{ks+j_1-kl} \prod_{j=0}^{m} \prod_{n=1}^{\frac{k}{2}} \frac{v_{klj+ks+j_1-l(2n-1)}}{u_{klj+ks+j_1-2l(n-1)}},$$

$$y_{klm+ks+j_1} = y_{ks+j_1-kl} \prod_{j=0}^{m} \prod_{n=1}^{\frac{k}{2}} \frac{u_{klj+ks+j_1-l(2n-1)}}{v_{klj+ks+j_1-2l(n-1)}} \qquad (2.11)$$

for $ks + j_1 \in \{kl - k - l, kl - k - l + 1, \ldots, 2kl - k - l - 1\}$, if k is even. Consequently, the solution in the closed form of system (1.6) follows from (2.5) and (2.10) or (2.11).

Now we give the solution form of system (1.6) when all the coefficients in system (1.6) are constant. To do this, we suppose that $a_n = a, b_n = b, \alpha_n = \alpha, \beta_n = \beta$ for every $n \in \mathbb{N}_0$. Then system (1.6) becomes

$$x_n = \frac{x_{n-k}y_{n-k-l}}{y_{n-l}\left(a + bx_{n-k}y_{n-k-l}\right)}, \; y_n = \frac{y_{n-k}x_{n-k-l}}{x_{n-l}\left(\alpha + \beta y_{n-k}x_{n-k-l}\right)}, \; n \in \mathbb{N}_0. \qquad (2.12)$$

Then we may assume that $gcd(k,l) = 1$. Indeed, if $|gcd(k,l)| = f > 1$, denoting the greatest common divisor of natural numbers k and l, then $k = fk_1$ and $l = fl_1$ for some $k_1, l_1 \in \mathbb{N}$ such that $gcd(k_1, l_1) = 1$. Since every $n \in \mathbb{N}_0$ has the form $n = mf + i$, for some $m \in \mathbb{N}_0$ and $i \in \{0, 1, \ldots, f - 1\}$, from (2.12) we get

$$x_{mf+i} = \frac{x_{f(m-k_1)+i}y_{f(m-k_1-l_1)+i}}{y_{f(m-l_1)+i}\left(a + bx_{f(m-k_1)+i}y_{f(m-k_1-l_1)+i}\right)}, \; y_{mf+i} = \frac{y_{f(m-k_1)+i}x_{f(m-k_1-l_1)+i}}{x_{f(m-l_1)+i}\left(\alpha + \beta y_{f(m-k_1)+i}x_{f(m-k_1-l_1)+i}\right)}. \qquad (2.13)$$

The change of variables

$$x_m^{(i)} = x_{mf+i}, \; y_m^{(i)} = y_{mf+i}, \; m \in \mathbb{N}_0, \; i \in \{0, 1, \ldots, f - 1\}$$

in (2.13) yields $\left(x_m^{(i)}\right)_{m \geq -(k_1+l_1)}$, $\left(y_m^{(i)}\right)_{m \geq -(k_1+l_1)}$, $i \in \{0, 1, \ldots, f - 1\}$, which are r independent solutions of the system

$$x_m^{(i)} = \frac{x_{m-k_1}^{(i)}y_{m-k_1-l_1}^{(i)}}{y_{m-l_1}^{(i)}\left(a + bx_{m-k_1}^{(i)}y_{m-k_1-l_1}^{(i)}\right)}, \; y_m^{(i)} = \frac{y_{m-k_1}^{(i)}x_{m-k_1-l_1}^{(i)}}{x_{m-l_1}^{(i)}\left(\alpha + \beta y_{m-k_1}^{(i)}x_{m-k_1-l_1}^{(i)}\right)}. \qquad (2.14)$$

Note that one can get system (2.13) by taking k_1 and l_1, respectively, instead of k and l in system (2.12). From now on, we assume that the greatest common divisor of k and l is equal to 1; that is, $gcd(k,l) = 1$. By substituting the formulas (2.6)–(2.7) into (2.10)–(2.11), we obtain the formulas for well-defined solutions of system (2.12) when $gcd(k,l) = 1$.

Theorem 2.1 *Assume that $a_n \neq 0,\ b_n \neq 0,\ \alpha_n \neq 0,\ \beta_n \neq 0,\ n \in \mathbb{N}_0$. Then the forbidden set of the initial values for system (1.6) is given by the set*

$$\mathcal{F} = \bigcup_{m \in \mathbb{N}_0} \bigcup_{i=0}^{k-1} \left\{ (x_{-k-l}, \ldots, x_{-1}, y_{-k-l}, \ldots, y_{-1}) \in \mathbb{R}^{2(k+l)} : \ x_{i-k}y_{i-k-l} = \frac{1}{c_m}, y_{i-k}x_{i-k-l} = \frac{1}{d_m}, where \right.$$

$$c_m := -\sum_{j=0}^{m} \frac{b_{kj+i}}{a_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{a_{kl+i}} \neq 0,\ d_m := -\sum_{j=0}^{m} \frac{\beta_{kj+i}}{\alpha_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{\alpha_{kl+i}} \neq 0 \left. \right\} \bigcup$$

$$\bigcup_{j=1}^{k+l} \left\{ (x_{-k-l}, \ldots, x_{-1}, y_{-k-l}, \ldots, y_{-1}) \in \mathbb{R}^{2(k+l)} : x_{-j} = 0,\ y_{-j} = 0 \right\}. \tag{2.15}$$

Proof   At the beginning of Section 2, we have acquired that the set

$$\bigcup_{j=1}^{k+l} \left\{ (x_{-k-l}, \ldots, x_{-1}, y_{-k-l}, \ldots, y_{-1}) \in \mathbb{R}^{2(k+l)} : x_{-j} = 0,\ y_{-j} = 0 \right\}$$

belongs to the forbidden set of the initial values for system (1.6). Now we assume that $x_n \neq 0$ and $y_n \neq 0$. Note that system (1.6) is undefined when the conditions $a_n + b_n x_{n-k} y_{n-k-l} = 0$ or $\alpha_n + \beta_n y_{n-k} x_{n-k-l} = 0$, that is, $x_{n-k} y_{n-k-l} = -\frac{a_n}{b_n}$ or $y_{n-k} x_{n-k-l} = -\frac{\alpha_n}{\beta_n}$, for some $n \in \mathbb{N}_0$, are satisfied (here we consider that $b_n \neq 0$ and $\beta_n \neq 0$ for every $n \in \mathbb{N}_0$). From this and the substitution $u_n = \frac{1}{x_n y_{n-l}},\ v_n = \frac{1}{y_n x_{n-l}}$, we get

$$u_{k(m-1)+i} = -\frac{b_{km+i}}{a_{km+i}},\ v_{k(m-1)+i} = -\frac{\beta_{km+i}}{\alpha_{km+i}} \tag{2.16}$$

for some $m \in \mathbb{N}_0$ and $i \in \{0, 1, \ldots, k-1\}$. Hence, we can determine the forbidden set of the initial values for system (1.6) by using the substitution $u_n = \frac{1}{x_n y_{n-l}},\ v_n = \frac{1}{y_n x_{n-l}}$. Now we consider the functions

$$f_{km+i}(t) := a_{km+i}t + b_{km+i},\ g_{km+i}(t) := \alpha_{km+i}t + \beta_{km+i},\ m \in \mathbb{N}_0,\ i \in \{0, 1, \ldots, k-1\}, \tag{2.17}$$

which correspond to the equations of (2.3). From (2.16) and (2.17), we can write

$$u_{km+i} = f_{km+i} \circ f_{k(m-1)+i} \circ \cdots \circ f_i (u_{i-k}), \tag{2.18}$$

$$v_{km+i} = g_{km+i} \circ g_{k(m-1)+i} \circ \cdots \circ g_i (v_{i-k}), \tag{2.19}$$

where $m \in \mathbb{N}_0$, and $i \in \{0, 1, \ldots, k-1\}$. By using (2.16) and implicit forms of (2.18)–(2.19), and considering $f_{km+i}^{-1}(0) = -\frac{b_{km+i}}{a_{km+i}},\ g_{km+i}^{-1}(0) = -\frac{\beta_{km+i}}{\alpha_{km+i}}$, for $m \in \mathbb{N}_0$ and $i \in \{0, 1, \ldots, k-1\}$, we have

$$u_{i-k} = f_i^{-1} \circ \cdots \circ f_{km+i}^{-1}(0),\ v_{i-k} = g_i^{-1} \circ \cdots \circ g_{km+i}^{-1}(0), \tag{2.20}$$

where $f_{km+i}^{-1}(t) = \frac{t-b_{km+i}}{a_{km+i}},\ g_{km+i}^{-1}(t) = \frac{t-\beta_{km+i}}{\alpha_{km+i}},\ m \in \mathbb{N}_0,\ i \in \{0, 1, \ldots, k-1\}$. From (2.20), we obtain

$$u_{i-k} = -\sum_{j=0}^{m} \frac{b_{kj+i}}{a_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{a_{kl+i}},\ v_{i-k} = -\sum_{j=0}^{m} \frac{\beta_{kj+i}}{\alpha_{kj+i}} \prod_{l=0}^{j-1} \frac{1}{\alpha_{kl+i}}$$

for some $m \in \mathbb{N}_0$ and $i \in \{0, 1, \ldots, k-1\}$. This means that if one of the conditions in (2.20) holds, then the mth iteration or $(m+1)$th iteration in system (1.6) can not be calculated. $\qquad\square$

3. The study of condition $k = 2,\ l = k$

In this section we give the asymptotic behavior of the solutions of system (2.12) when $k = 2, l = k$. In this case, the system becomes

$$x_n = \frac{x_{n-2}y_{n-k-2}}{y_{n-k}\left(a + bx_{n-2}y_{n-k-2}\right)},\ y_n = \frac{y_{n-2}x_{n-k-2}}{x_{n-k}\left(\alpha + \beta y_{n-2}x_{n-k-2}\right)},\ n \in \mathbb{N}_0. \tag{3.1}$$

In (2.11), if we employ the formulas given in (2.6) and (2.7) for the case $k = 2,\ l = k$, then the solution of system (3.1) is given by

$$x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{(1-a)\left(\beta + \alpha^{p(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-\alpha)-\beta\right)\right)}{(1-\alpha)\left(b + a^{p(2t+1)+s+1}\left(u_{j_1-2}(1-a)-b\right)\right)}, \tag{3.2}$$

$$y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{(1-\alpha)\left(b + a^{p(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a)-b\right)\right)}{(1-a)\left(\beta + \alpha^{p(2t+1)+s+1}\left(v_{j_1-2}(1-\alpha)-\beta\right)\right)}, \tag{3.3}$$

if $a \neq 1 \neq \alpha$,

$$x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{(1-a)\left(v_{-1-j_1} + \beta(p\left(2t+1\right)+s-t+j_1)\right)}{b + a^{p(2t+1)+s+1}\left(u_{j_1-2}(1-a)-b\right)}, \tag{3.4}$$

$$y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{b + a^{p(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a)-b\right)}{(1-a)\left(v_{j_1-2} + \beta(p\left(2t+1\right)+s+1)\right)}, \tag{3.5}$$

if $a \neq 1,\ \alpha = 1$,

$$x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{\beta + \alpha^{p(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-\alpha)-\beta\right)}{(1-\alpha)\left(u_{j_1-2} + b(p\left(2t+1\right)+s+1)\right)}, \tag{3.6}$$

$$y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{(1-\alpha)\left(u_{-1-j_1} + b(p\left(2t+1\right)+s-t+j_1)\right)}{\beta + \alpha^{p(2t+1)+s+1}\left(v_{j_1-2}(1-\alpha)-\beta\right)}, \tag{3.7}$$

if $a = 1,\ \alpha \neq 1$, and

$$x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{v_{-1-j_1} + \beta\left(p\left(2t+1\right)+s-t+j_1\right)}{u_{j_1-2} + b\left(p\left(2t+1\right)+s+1\right)}, \tag{3.8}$$

$$y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{u_{-1-j_1} + b\left(p\left(2t+1\right)+s-t+j_1\right)}{v_{j_1-2} + \beta\left(p\left(2t+1\right)+s+1\right)}, \tag{3.9}$$

if $a = 1 = \alpha$, where $m, t \in \mathbb{N}_0$, $j_1 \in \{0,1\}$ and $2s + j_1 \in \{k-2, k-1, ..., 3k-3\}$.

First, to present the $2k$- and $4k$-periodic solutions of system (3.1), we will give the following lemma.

Lemma 3.1 *Consider system (3.1). Then the next statements are true.*

(a) If $a \neq 1 \neq \alpha$, $b \neq 0 \neq \beta$, and $(1-a)\beta = b(1-\alpha)$ then system (3.1) has $2k$-periodic solutions.

(b) If $a \neq 1 \neq \alpha$, $b \neq 0 \neq \beta$, and $(1-a)\beta = -b(1-\alpha)$ then system (3.1) has $4k$-periodic solutions.

Proof Let

$$z_n = x_{n-2}y_{n-k-2} \text{ and } w_n = y_{n-2}x_{n-k-2}, \ n \in \mathbb{N}_0.$$

Then from (3.1) we have

$$z_{n+2} = \frac{z_n}{a + bz_n} \text{ and } w_{n+2} = \frac{w_n}{\alpha + \beta w_n}, \ n \in \mathbb{N}_0. \tag{3.10}$$

If $b \neq 0 \neq \beta$, then system (3.10) has a unique equilibrium solution $(\bar{z}, \bar{w})$, which is different from $(0,0)$; that is,

$$z_n = \bar{z} = \frac{1-a}{b} \neq 0, \quad w_n = \bar{w} = \frac{1-\alpha}{\beta} \neq 0, \ n \in \mathbb{N}_0.$$

If $\bar{z} = 0$ or $\bar{w} = 0$, then system (3.1) does not have a well-defined solution. From (3.10), we get that

$$x_{n-2} = \frac{1-a}{by_{n-k-2}} = \frac{(1-a)\beta}{b(1-\alpha)}x_{n-2k-2}, \text{ and } y_{n-2} = \frac{1-\alpha}{\beta y_{n-k-2}} = \frac{(1-\alpha)b}{\beta(1-a)}y_{n-2k-2}, \ n \geq k, \tag{3.11}$$

from which, along with the assumptions in (a)–(b), the results can be easily seen. $\square$

Now we investigate the asymptotic behavior of well-defined solutions of system (3.1) in detail according to the case of coefficients a and α.

3.1. Case $a \neq 1 \neq \alpha$

Here we describe the asymptotic behavior of the solution of system (3.1) for the case $a \neq 1 \neq \alpha$, by employing the next notation:

$$K_{j_1} := \frac{v_{-1-j_1}(1-a) - \beta}{a^{t+1-j_1}(u_{j_1-2}(1-a) - b)}, \quad L_{j_1} := \frac{u_{-1-j_1}(1-a) - b}{a^{t+1-j_1}(v_{j_1-2}(1-a) - \beta)},$$

$$\widehat{K}_{j_1} := \frac{(1-a)\left((-1)^{s-t+j_1}(v_{-1-j_1}(1+a) - \beta)\right)}{a^{t+1-j_1}(1+a)(u_{j_1-2}(1-a) - b)}, \quad \widehat{L}_{j_1} := \frac{(1+a)(u_{-1-j_1}(1-a) - b)}{a^{t+1-j_1}(1-a)\left((-1)^{s+1}(v_{j_1-2}(1+a) - \beta)\right)},$$

where $k \in \mathbb{N}$, $m, t \in \mathbb{N}_0$, $t = \left[\frac{k}{2}\right]$, $j_1 \in \{0, 1\}$.

Theorem 3.2 *Suppose that $a \neq 1 \neq \alpha$, $b \neq 0 \neq \beta$ and that $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements are true.*

(a) *If $|\alpha| > \max\{|a|, 1\}$ and $v_{j_1-2} \neq \beta/(1-\alpha) \neq v_{-1-j_1}$, for some $j_1 \in \{0, 1\}$, then $|x_n| \to \infty$, $y_n \to 0$, as $n \to \infty$.*

(b) *If $|a| > \max\{|\alpha|, 1\}$ and $u_{j_1-2} \neq b/(1-a) \neq u_{-1-j_1}$, for some $j_1 \in \{0, 1\}$, then $x_n \to 0$, $|y_n| \to \infty$, as $n \to \infty$.*

(c) *If* $max\{|a|,|\alpha|\} < 1$ *or* $u_{j_1-2} = b/(1-a) = u_{-1-j_1}$ *and* $v_{j_1-2} = \beta/(1-\alpha) = v_{-1-j_1}$, *and* $|(1-a)\beta| < |(1-\alpha)b|$, *for some* $j_1 \in \{0,1\}$, *then* $x_n \to 0$ *and* $|y_n| \to \infty$, *as* $n \to \infty$.

(d) *If* $max\{|a|,|\alpha|\} < 1$ *or* $u_{j_1-2} = b/(1-a) = u_{-1-j_1}$ *and* $v_{j_1-2} = \beta/(1-\alpha) = v_{-1-j_1}$, *and* $|(1-a)\beta| > |(1-\alpha)b|$, *for some* $j_1 \in \{0,1\}$, *then* $|x_n| \to \infty$ *and* $y_n \to 0$, *as* $n \to \infty$.

(e) *If* $max\{|a|,|\alpha|\} < 1$ *and* $(1-a)\beta = (1-\alpha)b$, *then* $(x_n,y_n)_{n\geq-k-2}$ *converges to a not necessarily prime* $2k$-*periodic solution of system* (3.1).

(f) *If* $u_{j_1-2} = b/(1-a) = u_{-1-j_1}$, $v_{j_1-2} = \beta/(1-\alpha) = v_{-1-j_1}$, *and* $(1-a)\beta = (1-\alpha)b$, *for some* $j_1 \in \{0,1\}$, *then* $(x_n,y_n)_{n\geq-k-2}$ *is a not necessarily prime* $2k$-*periodic solution of system* (3.1).

(g) *If* $max\{|a|,|\alpha|\} < 1$ *and* $(1-a)\beta = -(1-\alpha)b$, *then* $(x_n,y_n)_{n\geq-k-2}$ *converges to a not necessarily prime* $4k$-*periodic solution of system* (3.1).

(h) *If* $u_{j_1-2} = b/(1-a) = u_{-1-j_1}$, $v_{j_1-2} = \beta/(1-\alpha) = v_{-1-j_1}$, *and* $(1-a)\beta = -(1-\alpha)b$, *for some* $j_1 \in \{0,1\}$, *then* $(x_n,y_n)_{n\geq-k-2}$ *is a not necessarily prime* $4k$-*periodic solution of system* (3.1).

(i) *If* $a = \alpha$, $|a| > 1$ *and* $|K_{j_1}| < 1$, *for some* $j_1 \in \{0,1\}$, *then* $x_n \to 0$, *as* $n \to \infty$.

(j) *If* $a = \alpha$, $|a| > 1$ *and* $|L_{j_1}| < 1$, *for some* $j_1 \in \{0,1\}$, *then* $y_n \to 0$, *as* $n \to \infty$.

(k) *If* $a = \alpha$, $|a| > 1$ *and* $|K_{j_1}| > 1$, *for some* $j_1 \in \{0,1\}$, *then* $|x_n| \to \infty$, *as* $n \to \infty$.

(l) *If* $a = \alpha$, $|a| > 1$ *and* $|L_{j_1}| > 1$, *for some* $j_1 \in \{0,1\}$, *then* $|y_n| \to \infty$, *as* $n \to \infty$.

(m) *If* $a = \alpha$, $|a| > 1$ *and* $K_{j_1} = 1$, *for some* $j_1 \in \{0,1\}$, *then the sequence* $x_{2(2t+1)m+2s+j_1}$, *for* $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, *is convergent.*

(n) *If* $a = \alpha$, $|a| > 1$ *and* $L_{j_1} = 1$, *for some* $j_1 \in \{0,1\}$, *then the sequence* $y_{2(2t+1)m+2s+j_1}$, *for* $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, *is convergent.*

(o) *If* $a = \alpha$, $|a| > 1$ *and* $K_{j_1} = -1$, *for some* $j_1 \in \{0,1\}$, *then the sequences* $x_{4(2t+1)m+2s+j_1}, x_{4(2t+1)m+2(2t+1)+2s+j_1}$, *for* $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, *are convergent.*

(p) *If* $a = \alpha$, $|a| > 1$ *and* $L_{j_1} = -1$, *for some* $j_1 \in \{0,1\}$, *then the sequences* $y_{4(2t+1)m+2s+j_1}, y_{4(2t+1)m+2(2t+1)+2s+j_1}$, *for* $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, *are convergent.*

(q) *If* $a = -\alpha$, $|a| > 1$, m *is even, and* $|\widehat{K}_{j_1}| < 1$, *for some* $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, *then* $x_{4(2t+1)m+2s+j_1} \to 0$, *as* $m \to \infty$.

(r) *If* $a = -\alpha$, $|a| > 1$, m *is even, and* $|\widehat{L}_{j_1}| < 1$, *for some* $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, *then* $y_{4(2t+1)m+2s+j_1} \to 0$, *as* $m \to \infty$.

(s) *If* $a = -\alpha$, $|a| > 1$, m *is even, and* $|\widehat{K}_{j_1}| > 1$, *for some* $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, *then* $|x_{4(2t+1)m+2s+j_1}| \to \infty$, *as* $m \to \infty$.

(t) If $a = -\alpha$, $|a| > 1$, m is even, and $|\widehat{L}_{j_1}| > 1$, for some $j_1 \in \{0, 1\}$, $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then $|y_{4(2t+1)m+2s+j_1}| \to \infty$, as $m \to \infty$.

(u) If $a = -\alpha$, $|a| > 1$, m is even, and $\widehat{K}_{j_1} = 1$, for some $j_1 \in \{0, 1\}$, $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then the sequence $x_{4(2t+1)m+2s+j_1}$ is convergent.

(v) If $a = -\alpha$, $|a| > 1$, m is even, and $\widehat{L}_{j_1} = 1$, for some $j_1 \in \{0, 1\}$, $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then the sequence $y_{4(2t+1)m+2s+j_1}$ is convergent.

(w) If $a = -\alpha$, $|a| > 1$, m is even, and $\widehat{K}_{j_1} = -1$, for some $j_1 \in \{0, 1\}$, $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then the sequences $x_{8(2t+1)m+2s+j_1}$ and $x_{8(2t+1)m+4(2t+1)+2s+j_1}$ are convergent.

(x) If $a = -\alpha$, $|a| > 1$, m is even, and $\widehat{L}_{j_1} = -1$, for some $j_1 \in \{0, 1\}$, $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then the sequences $y_{8(2t+1)m+2s+j_1}$ and $y_{8(2t+1)m+4(2t+1)+2s+j_1}$ are convergent.

Proof Let

$$p_m^{t,2s+j_1} = \frac{(1-a)\left((v_{-1-j_1}(1-\alpha) - \beta)\alpha^{m(2t+1)+s-t+j_1} + \beta\right)}{(1-\alpha)\left((u_{j_1-2}(1-a) - b)a^{m(2t+1)+s+1} + b\right)}, \tag{3.12}$$

for $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$, and

$$r_m^{t,2s+j_1} = \frac{(1-\alpha)\left((u_{-1-j_1}(1-a) - b)a^{m(2t+1)+s-t+j_1} + b\right)}{(1-a)\left((v_{j_1-2}(1-\alpha) - \beta)\alpha^{m(2t+1)+s+1} + \beta\right)}, \tag{3.13}$$

for $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$.
(a): Note that in this case

$$\lim_{m \to \infty} |p_m^{t,2s+j_1}| = \infty, \quad \lim_{m \to \infty} r_m^{t,2s+j_1} = 0,$$

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\}$, from which, along with formulas (3.2) and (3.3), the result can be seen easily.
(b): Note that in this case

$$\lim_{m \to \infty} p_m^{t,2s+j_1} = 0, \quad \lim_{m \to \infty} |r_m^{t,2s+j_1}| = \infty,$$

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\}$, from which, along with formulas (3.2) and (3.3), the result can be obtained easily.
(c)–(d): In this case we get

$$\lim_{m \to \infty} |p_m^{t,2s+j_1}| = \left|\frac{(1-a)\beta}{(1-\alpha)b}\right|, \quad \lim_{m \to \infty} |r_m^{t,2s+j_1}| = \left|\frac{(1-\alpha)b}{(1-a)\beta}\right|,$$

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\}$, from which, along with the assumptions in (c) and (d), the statements easily follow.

(e): Employing the Taylor expansion for $(1+x)^{-1}$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, we have, for sufficiently large m and each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, \ j_1 \in \{0, 1\}$,

$$
\begin{aligned}
p_m^{t, 2s+j_1} &= \frac{(1-a)\left(\beta + \alpha^{m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-\alpha) - \beta\right)\right)}{(1-\alpha)\left(b + a^{m(2t+1)+s+1}\left(u_{j_1-2}(1-a) - b\right)\right)} \\[2mm]
&= \frac{(1-a)\beta\left(1 + \alpha^{m(2t+1)+s-t+j_1}\frac{\left(v_{-1-j_1}(1-\alpha)-\beta\right)}{\beta}\right)}{(1-\alpha)b\left(1 + a^{m(2t+1)+s+1}\frac{\left(u_{j_1-2}(1-a)-b\right)}{b}\right)} \\[2mm]
&= 1 + \alpha^{m(2t+1)+s-t+j_1}\frac{\left(v_{-1-j_1}(1-\alpha)-\beta\right)}{\beta} - a^{m(2t+1)+s+1}\frac{\left(u_{j_1-2}(1-a)-b\right)}{b} \\[2mm]
&\quad + \mathcal{O}\left(a^{2m(2t+1)}\right),
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
r_m^{t, 2s+j_1} &= \frac{(1-\alpha)\left(b + a^{m(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a) - b\right)\right)}{(1-a)\left(\beta + \alpha^{m(2t+1)+s+1}\left(v_{j_1-2}(1-\alpha) - \beta\right)\right)} \\[2mm]
&= \frac{(1-\alpha)b\left(1 + a^{m(2t+1)+s-t+j_1}\frac{\left(u_{-1-j_1}(1-a)-b\right)}{b}\right)}{(1-a)\beta\left(1 + \alpha^{m(2t+1)+s+1}\frac{\left(v_{j_1-2}(1-\alpha)-\beta\right)}{\beta}\right)} \\[2mm]
&= 1 + a^{m(2t+1)+s-t+j_1}\frac{\left(u_{-1-j_1}(1-a)-b\right)}{b} - \alpha^{m(2t+1)+s+1}\frac{\left(v_{j_1-2}(1-\alpha)-\beta\right)}{\beta} \\[2mm]
&\quad + \mathcal{O}\left(\alpha^{2m(2t+1)}\right).
\end{aligned}
\tag{3.15}
$$

Employing (3.14)–(3.15) in (3.2)–(3.3) and the assumption $\max\{|a|, |\alpha|\} < 1$, the convergence of the sequences $x_{2(2t+1)m+2s+j_1}$ and $y_{2(2t+1)m+2s+j_1}$ for $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0, 1\}$, can be seen easily and consequently $(x_n, y_n)_{n \geq -k-2}$ converges to a not necessarily prime $2k$-periodic solution of system (3.1).

(f): From (3.2) and (3.3), we get

$$
p_m^{t, 2s+j_1} = \frac{(1-a)\beta}{(1-\alpha)b} = 1, \quad r_m^{t, 2s+j_1} = \frac{(1-\alpha)b}{(1-a)\beta} = 1,
$$

for every $m \in \mathbb{N}_0$, if $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, \ j_1 \in \{0, 1\}$. That is, $x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)}$ and $y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)}$, for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0, 1\}$, from which the result can be obtained easily.

(g): From (3.12) and (3.13), in this case, we have

$$
\begin{aligned}
p_m^{t,2s+j_1} &= \frac{(1-a)\left(\beta + \alpha^{m(2t+1)+s-t+j_1}(v_{-1-j_1}(1-\alpha)-\beta)\right)}{(1-\alpha)\left(b + a^{m(2t+1)+s+1}(u_{j_1-2}(1-a)-b)\right)} \\
&= \frac{(1-a)\beta\left(1 + \alpha^{m(2t+1)+s-t+j_1}\frac{(v_{-1-j_1}(1-\alpha)-\beta)}{\beta}\right)}{(1-\alpha)b\left(1 + a^{m(2t+1)+s+1}\frac{(u_{j_1-2}(1-a)-b)}{b}\right)} \\
&= -\left(1 + \alpha^{m(2t+1)+s-t+j_1}\frac{(v_{-1-j_1}(1-\alpha)-\beta)}{\beta} - a^{m(2t+1)+s+1}\frac{(u_{j_1-2}(1-a)-b)}{b}\right. \\
&\quad + \left.\mathcal{O}\left(a^{2m(2t+1)}\right)\right),
\end{aligned}
\tag{3.16}
$$

for sufficiently large m if $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, $j_1 \in \{0,1\}$, and

$$
\begin{aligned}
r_m^{t,2s+j_1} &= \frac{(1-\alpha)\left(b + a^{m(2t+1)+s-t+j_1}(u_{-1-j_1}(1-a)-b)\right)}{(1-a)\left(\beta + \alpha^{m(2t+1)+s+1}(v_{j_1-2}(1-\alpha)-\beta)\right)} \\
&= \frac{(1-\alpha)b\left(1 + a^{m(2t+1)+s-t+j_1}\frac{(u_{-1-j_1}(1-a)-b)}{b}\right)}{(1-a)\beta\left(1 + \alpha^{m(2t+1)+s+1}\frac{(v_{j_1-2}(1-\alpha)-\beta)}{\beta}\right)} \\
&= -\left(1 + a^{m(2t+1)+s-t+j_1}\frac{(u_{-1-j_1}(1-a)-b)}{b} - \alpha^{m(2t+1)+s+1}\frac{(v_{j_1-2}(1-\alpha)-\beta)}{\beta}\right. \\
&\quad + \left.\mathcal{O}\left(\alpha^{2m(2t+1)}\right)\right),
\end{aligned}
\tag{3.17}
$$

for sufficiently large m if $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, $j_1 \in \{0,1\}$. Employing (3.16)–(3.17) in (3.2)–(3.3), Taylor expansion for $(1+x)^{-1}$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$ and the assumption $max\{|a|,|\alpha|\} < 1$, the convergence of the sequences $x_{4(2t+1)m+2s+j_1}, y_{4(2t+1)m+2s+j_1}$ for $2s+j_1 \in \{k-2,k-1,\ldots,5k-3\}, j_1 \in \{0,1\}$ easily follows, from which it follows that $(x_n, y_n)_{n \geq -k-2}$ converges to a not necessarily prime $4k$-periodic solution of system (3.1).

(h): From (3.12) and (3.13), in this case, we get

$$
p_m^{t,2s+j_1} = \frac{(1-a)\beta}{(1-\alpha)b} = -1, \quad r_m^{t,2s+j_1} = \frac{(1-\alpha)b}{(1-a)\beta} = -1,
$$

for every $m \in \mathbb{N}_0$, if $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}$, $j_1 \in \{0,1\}$. That is, $x_{2(2t+1)m+2s+j_1} = (-1)^{m+1}x_{2s+j_1-2(2t+1)}$ and $y_{2(2t+1)m+2s+j_1} = (-1)^{m+1}y_{2s+j_1-2(2t+1)}$ for each $2s+j_1 \in \{k-2,k-1,\ldots,3k-3\}, j_1 \in \{0,1\}$, from which the result can be obtained easily.

(i)-(l): From (3.12) and (3.13), in this case, we get

$$
\lim_{m\to\infty} p_m^{t,2s+j_1} = \lim_{m\to\infty} \frac{(v_{-1-j_1}(1-a)-\beta) + \frac{\beta}{a^{m(2t+1)+s-t+j_1}}}{(u_{j_1-2}(1-a)-b)a^{t+1-j_1} + \frac{b}{a^{m(2t+1)+s-t+j_1}}} = K_{j_1},
\tag{3.18}
$$

for $2s + j_1 \in \{k - 3, k - 2, \ldots, 3k - 4\}, \quad j_1 \in \{0, 1\},$

$$\lim_{m \to \infty} r_m^{t, 2s+j_1} = \lim_{m \to \infty} \frac{(u_{-1-j_1}(1 - a) - b) + \frac{b}{a^{m(2t+1)+s-t+j_1}}}{(v_{j_1-2}(1 - a) - \beta)a^{t+1-j_1} + \frac{\beta}{a^{m(2t+1)+s-t+j_1}}} = L_{j_1}, \tag{3.19}$$

for $2s + j_1 \in \{k - 3, k - 2, \ldots, 3k - 4\}, \quad j_1 \in \{0, 1\}$. Employing (3.18)–(3.19) in (3.2)–(3.3) and the assumptions in (i)–(l), the results easily follow.

(m): From (3.12) and (3.13), we have

$$\begin{aligned}
p_m^{t, 2s+j_1} &= \frac{(v_{-1-j_1}(1 - a) - \beta)a^{m(2t+1)+s-t+j_1} + \beta}{(u_{j_1-2}(1 - a) - b)a^{m(2t+1)+s+1} + b} \\[2mm]
&= \frac{(v_{-1-j_1}(1 - a) - \beta)a^{m(2t+1)+s-t+j_1}\left(1 + \frac{\beta}{a^{m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-a)-\beta\right)}\right)}{(u_{j_1-2}(1 - a) - b)a^{m(2t+1)+s+1}\left(1 + \frac{b}{a^{m(2t+1)+s+1}\left(u_{j_1-2}(1-a)-b\right)}\right)} \\[2mm]
&= 1 + \frac{\beta}{a^{m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1 - a) - \beta\right)} - \frac{b}{a^{m(2t+1)+s+1}\left(u_{j_1-2}(1 - a) - b\right)} \\[2mm]
&\quad + \mathcal{O}\left(\frac{1}{a^{2m(2t+1)}}\right),
\end{aligned} \tag{3.20}$$

for sufficiently large m if $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, \quad j_1 \in \{0, 1\}$, from which the convergence of the sequence $x_{2(2t+1)m+2s+j_1}$, for $j_1 \in \{0, 1\}$, can be seen easily.

(n): From (3.12) and (3.13), we get

$$\begin{aligned}
r_m^{t, 2s+j_1} &= \frac{(u_{-1-j_1}(1 - a) - b)a^{m(2t+1)+s-t+j_1} + b}{(v_{j_1-2}(1 - a) - \beta)a^{m(2t+1)+s+1} + \beta} \\[2mm]
&= \frac{(u_{-1-j_1}(1 - a) - b)a^{m(2t+1)+s-t+j_1}\left(1 + \frac{b}{a^{m(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a)-b\right)}\right)}{(v_{j_1-2}(1 - a) - \beta)a^{m(2t+1)+s+1}\left(1 + \frac{\beta}{a^{m(2t+1)+s+1}\left(v_{j_1-2}(1-a)-\beta\right)}\right)} \\[2mm]
&= 1 + \frac{b}{a^{m(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1 - a) - b\right)} - \frac{\beta}{a^{m(2t+1)+s+1}\left(v_{j_1-2}(1 - a) - \beta\right)} \\[2mm]
&\quad + \mathcal{O}\left(\frac{1}{a^{2m(2t+1)}}\right),
\end{aligned} \tag{3.21}$$

for sufficiently large m if $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, \quad j_1 \in \{0, 1\}$, from which the convergence of the sequence $y_{2(2t+1)m+2s+j_1}$, for $j_1 \in \{0, 1\}$, can be obtained easily.

(o): From (3.12), in this case, we have

$$
\begin{aligned}
p_m^{t,2s+j_1} &= \frac{(v_{-1-j_1}(1-a)-\beta)a^{m(2t+1)+s-t+j_1}+\beta}{(u_{j_1-2}(1-a)-b)a^{m(2t+1)+s+1}+b} \\[2mm]
&= \frac{(v_{-1-j_1}(1-a)-\beta)a^{m(2t+1)+s-t+j_1}\left(1+\frac{\beta}{a^{m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-a)-\beta\right)}\right)}{(u_{j_1-2}(1-a)-b)a^{m(2t+1)+s+1}\left(1+\frac{b}{a^{m(2t+1)+s+1}\left(u_{j_1-2}(1-a)-b\right)}\right)} \\[2mm]
&= -\left(1+\frac{\beta}{a^{m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-a)-\beta\right)}-\frac{b}{a^{m(2t+1)+s+1}\left(u_{j_1-2}(1-a)-b\right)}\right. \\[2mm]
&\quad + \left.\mathcal{O}\left(\frac{1}{a^{2m(2t+1)}}\right)\right),
\end{aligned} \tag{3.22}
$$

for sufficiently large m if $2s+j_1\in\{k-2,k-1,\ldots,3k-3\}$, $j_1\in\{0,1\}$, from which the convergence of the sequences $x_{4(2t+1)m+2s+j_1}, x_{4(2t+1)m+2(2t+1)+2s+j_1}$ for $j_1\in\{0,1\}$ easily follows.

(p): From (3.13), in this case, we obtain

$$
\begin{aligned}
r_m^{t,2s+j_1} &= \frac{(u_{-1-j_1}(1-a)-b)a^{m(2t+1)+s-t+j_1}+b}{(v_{j_1-2}(1-a)-\beta)a^{m(2t+1)+s+1}+\beta} \\[2mm]
&= \frac{(u_{-1-j_1}(1-a)-b)a^{m(2t+1)+s-t+j_1}\left(1+\frac{b}{a^{m(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a)-b\right)}\right)}{(v_{j_1-2}(1-a)-\beta)a^{m(2t+1)+s+1}\left(1+\frac{\beta}{a^{m(2t+1)+s+1}\left(v_{j_1-2}(1-a)-\beta\right)}\right)} \\[2mm]
&= -\left(1+\frac{b}{a^{m(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a)-b\right)}-\frac{\beta}{a^{m(2t+1)+s+1}\left(v_{j_1-2}(1-a)-\beta\right)}\right. \\[2mm]
&\quad + \left.\mathcal{O}\left(\frac{1}{a^{2m(2t+1)}}\right)\right),
\end{aligned} \tag{3.23}
$$

for sufficiently large m if $2s+j_1\in\{k-2,k-1,\ldots,3k-3\}$, $j_1\in\{0,1\}$. Using (3.23) in (3.3) and the Taylor expansion for $(1+x)^{-1}$ on the interval $(-\epsilon,\epsilon)$, where $\epsilon>0$, the convergence of the sequences $y_{4(2t+1)m+2s+j_1}$, $y_{4(2t+1)m+2(2t+1)+2s+j_1}$ for $j_1\in\{0,1\}$ easily follows.

(q)–(t): To obtain these four results it is enough to note that the limits of the general terms (3.12)–(3.13) in (3.2)–(3.3) are equal to $\widehat{K}_{j_1}$ and $\widehat{L}_{j_1}$, respectively, and then employing the assumptions in (t)–(q), the results can be seen easily.

(u): From (3.12), in this case, we get

$$
\begin{aligned}
p_{2m}^{t,2s+j_1} &= \frac{1+\frac{\beta}{(-a)^{2m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1+a)-\beta\right)}}{1+\frac{b}{a^{m(2t+1)+s+1}\left(u_{j_1-2}(1-a)-b\right)}} \\[2mm]
&= 1+\frac{\beta}{(-a)^{2m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1+a)-\beta\right)}-\frac{b}{a^{2m(2t+1)+s+1}\left(u_{j_1-2}(1-a)-b\right)} \\[2mm]
&\quad + \mathcal{O}\left(\frac{1}{a^{4m(2t+1)}}\right),
\end{aligned} \tag{3.24}
$$

for sufficiently large m if $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$, from which the convergence of the sequence $x_{4(2t+1)m+2s+j_1}$ for $j_1 \in \{0, 1\}$ easily follows.

(v): From (3.13), in this case, we have

$$
\begin{aligned}
r_{2m}^{t,2s+j_1} &= \frac{1 + \frac{b}{a^{2m(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a)-b\right)}}{1 + \frac{\beta}{(-a)^{2m(2t+1)+s+1}\left(v_{j_1-2}(1+a)-\beta\right)}} \\
&= 1 + \frac{b}{a^{2m(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a)-b\right)} - \frac{\beta}{(-a)^{2m(2t+1)+s+1}\left(v_{j_1-2}(1+a)-\beta\right)} \\
&\quad + \mathcal{O}\left(\frac{1}{a^{4m(2t+1)}}\right),
\end{aligned}
\tag{3.25}
$$

for sufficiently large m if $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$, from which the convergence of the sequence $y_{4(2t+1)m+2s+j_1}$ for $j_1 \in \{0, 1\}$ can be seen easily.

(w): From (3.12), in this case, we get

$$
\begin{aligned}
p_{2m}^{t,2s+j_1} &= \frac{(1-a)(v_{-1-j_1}(1+a)-\beta)(-a)^{2m(2t+1)+s-t+j_1}+\beta}{(1+a)(u_{j_1-2}(1-a)-b)a^{2m(2t+1)+s+1}+b} \\
&= \frac{(1-a)(v_{-1-j_1}(1+a)-\beta)(-a)^{2m(2t+1)+s-t+j_1}\left(1+\frac{\beta}{(-a)^{2m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1+a)-\beta\right)}\right)}{(1+a)(u_{j_1-2}(1-a)-b)a^{2m(2t+1)+s+1}\left(1+\frac{b}{a^{2m(2t+1)+s+1}\left(u_{j_1-2}(1-a)-b\right)}\right)} \\
&= -\left(1 + \frac{\beta}{(-a)^{2m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1+a)-\beta\right)} - \frac{b}{a^{2m(2t+1)+s+1}\left(u_{j_1-2}(1-a)-b\right)}\right. \\
&\quad \left. + \mathcal{O}\left(\frac{1}{a^{4m(2t+1)}}\right)\right),
\end{aligned}
\tag{3.26}
$$

for sufficiently large m if $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, $j_1 \in \{0, 1\}$. Using (3.26) in (3.2) and the Taylor expansion for $(1+x)^{-1}$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, the convergence of the sequences $x_{8(2t+1)m+2s+j_1}$, $x_{8(2t+1)m+4(2t+1)+2s+j_1}$, for $j_1 \in \{0, 1\}$, can be seen easily.

(x): From (3.13), in this case, we have

$$
\begin{aligned}
r_{2m}^{t,2s+j_1} &= \frac{(1+a)(u_{-1-j_1}(1-a)-b)a^{2m(2t+1)+s-t+j_1}+b}{(1-a)(v_{j_1-2}(1+a)-\beta)(-a)^{2m(2t+1)+s+1}+\beta} \\
&= \frac{(1+a)(u_{-1-j_1}(1-a)-b)a^{2m(2t+1)+s-t+j_1}\left(1+\frac{b}{a^{2m(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a)-b\right)}\right)}{(1-a)(v_{j_1-2}(1+a)-\beta)(-a)^{2m(2t+1)+s+1}\left(1+\frac{\beta}{(-a)^{2m(2t+1)+s+1}\left(v_{j_1-2}(1+a)-\beta\right)}\right)} \\
&= -\left(1 + \frac{b}{a^{2m(2t+1)+s-t+j_1}\left(u_{-1-j_1}(1-a)-b\right)} - \frac{\beta}{(-a)^{2m(2t+1)+s+1}\left(v_{j_1-2}(1+a)-\beta\right)}\right. \\
&\quad \left. + \mathcal{O}\left(\frac{1}{a^{4m(2t+1)}}\right)\right),
\end{aligned}
\tag{3.27}
$$

for sufficiently large m if $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, $j_1 \in \{0, 1\}$, from which the convergence of the sequences $y_{8(2t+1)m+2s+j_1}$, $y_{8(2t+1)m+4(2t+1)+2s+j_1}$, for $j_1 \in \{0, 1\}$, can be obtained easily.

$\square$

Theorem 3.3 *Assume that $a \neq 1 \neq \alpha$, $b \neq 0 \neq \beta$, $a = -\alpha$, $|a| > 1$, $m, t, w, h \in \mathbb{N}_0$, $t = \left[\frac{k}{2}\right]$, $j_1 \in \{0, 1\}$, m is odd, and $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements are true.*

(a) *If $\left| \frac{(1-a)(v_{-1-j_1}(1+a)-\beta)}{(1+a)a^{t+1-j_1}(u_{j_1-2}(1-a)-b)} \right| < 1$ for some $j_1 \in \{0, 1\}$, then $x_{4(2t+1)m+4t+2+4w+j_1} \to 0$,*

   $x_{4(2t+1)m+4t+4w+4+j_1} \to 0$ *as $m \to \infty$.*

(b) *If $\left| \frac{(1-a)(v_{-1-j_1}(1+a)-\beta)}{(1+a)a^{t+1-j_1}(u_{j_1-2}(1-a)-b)} \right| > 1$ for some $j_1 \in \{0, 1\}$, then $|x_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$,*

   $|x_{4(2t+1)m+4t+4w+4+j_1}| \to \infty$ *as $m \to \infty$.*

(c) *If $\frac{(1-a)(v_{-1-j_1}(1+a)-\beta)}{(1+a)a^{t+1-j_1}(u_{j_1-2}(1-a)-b)} = 1$ for some $j_1 \in \{0, 1\}$, then the sequences are $x_{4(2t+1)m+4t+2+4w+j_1}$,*

   $x_{4(2t+1)m+4t+4w+4+j_1}$ *are convergent.*

(d) *If $\frac{(1-a)(v_{-1-j_1}(1+a)-\beta)}{(1+a)a^{t+1-j_1}(u_{j_1-2}(1-a)-b)} = -1$ for some $j_1 \in \{0, 1\}$, then the sequences $x_{8(2t+1)m+4t+2+4w+j_1}$,*

   $x_{8(2t+1)m+12t+6+4w+j_1}$, $x_{8(2t+1)m+4t+4w+4+j_1}$, *and $x_{8(2t+1)m+12t+4w+8+j_1}$ are convergent.*

(e) *If $\left| \frac{(1+a)(u_{-1-j_1}(1-a)-b)}{(1-a)a^{t+1-j_1}(v_{j_1-2}(1+a)-\beta)} \right| < 1$ for some $j_1 \in \{0, 1\}$, then $y_{4(2t+1)m+4t+2+4w+j_1} \to 0$ as $m \to \infty$.*

(f) *If $\left| \frac{(1+a)(u_{-1-j_1}(1-a)-b)}{(1-a)a^{t+1-j_1}(v_{j_1-2}(1+a)-\beta)} \right| > 1$ for some $j_1 \in \{0, 1\}$, then $|y_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$ as $m \to \infty$.*

(g) *If $\frac{(1+a)(u_{-1-j_1}(1-a)-b)}{(1-a)a^{t+1-j_1}(v_{j_1-2}(1+a)-\beta)} = 1$ for some $j_1 \in \{0, 1\}$, then the sequence $y_{4(2t+1)m+4t+2+4w+j_1}$ is conver-*

   gent.

(h) *If $\frac{(1+a)(u_{-1-j_1}(1-a)-b)}{(1-a)a^{t+1-j_1}(v_{j_1-2}(1+a)-\beta)} = -1$ for some $j_1 \in \{0, 1\}$, then the sequences $y_{8(2t+1)m+4t+2+4w+j_1}$ and*

   $y_{8(2t+1)m+12t+6+4w+j_1}$ *are convergent.*

(i) *If $\left| \frac{(a-1)(v_{-1-j_1}(1+a)-\beta)}{(1+a)a^{t+1-j_1}(u_{j_1-2}(1-a)-b)} \right| < 1$ for some $j_1 \in \{0, 1\}$, then $x_{4(2t+1)m+4t+2+4w+j_1} \to 0$,*

   $x_{4(2t+1)m+4t+4w+4+j_1} \to 0$ *as $m \to \infty$.*

(j) *If $\left| \frac{(a-1)(v_{-1-j_1}(1+a)-\beta)}{(1+a)a^{t+1-j_1}(u_{j_1-2}(1-a)-b)} \right| > 1$ for some $j_1 \in \{0, 1\}$, then $|x_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$,*

   $|x_{4(2t+1)m+4t+4w+4+j_1}| \to \infty$ *as $m \to \infty$.*

(k) *If $\frac{(a-1)(v_{-1-j_1}(1+a)-\beta)}{(1+a)a^{t+1-j_1}(u_{j_1-2}(1-a)-b)} = 1$ for some $j_1 \in \{0, 1\}$, then the sequences $x_{4(2t+1)m+4t+2+4w+j_1}$,*

   $x_{4(2t+1)m+4t+4w+4+j_1}$ *are convergent.*

(l) If $\frac{(a-1)\left(v_{-1-j_1}(1+a)-\beta\right)}{(1+a)a^{t+1-j_1}\left(u_{j_1-2}(1-a)-b\right)} = -1$ for some $j_1 \in \{0,1\}$, then the sequences $x_{8(2t+1)m+4t+2+4w+j_1}$,

$x_{8(2t+1)m+12t+6+4w+j_1}$, $x_{8(2t+1)m+4t+4w+4+j_1}$, $x_{8(2t+1)m+12t+4w+8+j_1}$ are convergent.

(m) If $\left|\frac{(1+a)\left(u_{-1-j_1}(1-a)-b\right)}{(a-1)a^{t+1-j_1}\left(v_{j_1-2}(1+a)-\beta\right)}\right| < 1$ for some $j_1 \in \{0,1\}$, then $y_{4(2t+1)m+4t+4w+4+j_1} \to 0$ as $m \to \infty$.

(n) If $\left|\frac{(1+a)\left(u_{-1-j_1}(1-a)-b\right)}{(a-1)a^{t+1-j_1}\left(v_{j_1-2}(1+a)-\beta\right)}\right| > 1$ for some $j_1 \in \{0,1\}$, then $|y_{4(2t+1)m+4t+4w+4+j_1}| \to \infty$ as $m \to \infty$.

(o) If $\frac{(1+a)\left(u_{-1-j_1}(1-a)-b\right)}{(a-1)a^{t+1-j_1}\left(v_{j_1-2}(1+a)-\beta\right)} = 1$ for some $j_1 \in \{0,1\}$, then the sequence $y_{4(2t+1)m+4t+4w+4+j_1}$ is convergent.

(p) If $\frac{(1+a)\left(u_{-1-j_1}(1-a)-b\right)}{(a-1)a^{t+1-j_1}\left(v_{j_1-2}(1+a)-\beta\right)} = -1$ for some $j_1 \in \{0,1\}$, then the sequences $y_{8(2t+1)m+4t+4w+4+j_1}$ and

$y_{8(2t+1)m+12t+4w+8+j_1}$ are convergent.

Proof   Let $a = -\alpha$, $|a| > 1$, and m be odd. From (3.12)–(3.13) we get

$$\widehat{p}_{2m+1}^{s,s-t+j_1} = \frac{(1-a)\left((v_{-1-j_1}(1+a)-\beta)(-1)^{(2m+1)(2t+1)+s-t+j_1} + \frac{\beta}{a^{(2m+1)(2t+1)+s-t+j_1}}\right)}{(1+a)a^{t+1-j_1}\left((u_{j_1-2}(1-a)-b) + \frac{b}{a^{(2m+1)(2t+1)+s+1}}\right)}, \qquad (3.28)$$

$$\widehat{r}_{2m+1}^{s,s-t+j_1} = \frac{(1+a)\left((u_{-1-j_1}(1-a)-b) + \frac{b}{a^{(2m+1)(2t+1)+s-t+j_1}}\right)}{(1-a)a^{t+1-j_1}\left((v_{j_1-2}(1+a)-\beta)(-1)^{(2m+1)(2t+1)+s+1} + \frac{\beta}{a^{(2m+1)(2t+1)+s+1}}\right)}. \qquad (3.29)$$

Here, there are four cases to be considered.

- s **is even and** $s-t+j_1$ **is odd:** In this case we have

$$\widehat{p}_{2m+1}^{2w,2h+1} = \frac{(1-a)\left((v_{-1-j_1}(1+a)-\beta)(-1)^{(2m+1)(2t+1)+2h+1} + \frac{\beta}{a^{(2m+1)(2t+1)+2h+1}}\right)}{(1+a)a^{t+1-j_1}\left((u_{j_1-2}(1-a)-b) + \frac{b}{a^{(2m+1)(2t+1)+2w+1}}\right)}, \qquad (3.30)$$

$$\widehat{r}_{2m+1}^{2w,2h+1} = \frac{(1+a)\left((u_{-1-j_1}(1-a)-b) + \frac{b}{a^{(2m+1)(2t+1)+2h+1}}\right)}{(1-a)a^{t+1-j_1}\left((v_{j_1-2}(1+a)-\beta)(-1)^{(2m+1)(2t+1)+2w+1} + \frac{\beta}{a^{(2m+1)(2t+1)+2w+1}}\right)}, \qquad (3.31)$$

where $s = 2w$ and $s-t+j_1 = 2h+1$, for $w, h \in \mathbb{N}_0$. From (3.30)–(3.31), we have

$$\lim_{m\to\infty} \widehat{p}_{2m+1}^{2w,2h+1} = \frac{(1-a)\left(v_{-1-j_1}(1+a)-\beta\right)}{(1+a)a^{t+1-j_1}\left(u_{j_1-2}(1-a)-b\right)}, \qquad (3.32)$$

$$\lim_{m\to\infty} \widehat{r}_{2m+1}^{2w,2h+1} = \frac{(1+a)\left(u_{-1-j_1}(1-a)-b\right)}{(1-a)a^{t+1-j_1}\left(v_{j_1-2}(1+a)-\beta\right)}. \qquad (3.33)$$

- s **and** $s - t + j_1$ **are both even:** In this case we get

$$\widehat{p}_{2m+1}^{2w,2h} = \frac{(1-a)\left((v_{-1-j_1}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2h} + \frac{\beta}{a^{(2m+1)(2t+1)+2h}}\right)}{(1+a)a^{t+1-j_1}\left((u_{j_1-2}(1-a) - b) + \frac{b}{a^{(2m+1)(2t+1)+2w+1}}\right)}, \qquad (3.34)$$

$$\widehat{r}_{2m+1}^{2w,2h} = \frac{(1+a)\left((u_{-1-j_1}(1-a) - b) + \frac{b}{a^{(2m+1)(2t+1)+2h}}\right)}{(1-a)a^{t+1-j_1}\left((v_{j_1-2}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2w+1} + \frac{\beta}{a^{(2m+1)(2t+1)+2w+1}}\right)}, \qquad (3.35)$$

where $s = 2w$ and $s - t + j_1 = 2h$, for $w, h \in \mathbb{N}_0$. From (3.34)–(3.35), we obtain

$$\lim_{m \to \infty} \widehat{p}_{2m+1}^{2w,2h} = \frac{(a-1)\left(v_{-1-j_1}(1+a) - \beta\right)}{(1+a)a^{t+1-j_1}\left(u_{j_1-2}(1-a) - b\right)}, \qquad (3.36)$$

$$\lim_{m \to \infty} \widehat{r}_{2m+1}^{2w,2h} = \frac{(1+a)\left(u_{-1-j_1}(1-a) - b\right)}{(1-a)a^{t+1-j_1}\left(v_{j_1-2}(1+a) - \beta\right)}, \qquad (3.37)$$

- s **and** $s - t + j_1$ **are both odd:** In this case we have

$$\widehat{p}_{2m+1}^{2w+1,2h+1} = \frac{(1-a)\left((v_{-1-j_1}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2h+1} + \frac{\beta}{a^{(2m+1)(2t+1)+2h+1}}\right)}{(1+a)a^{t+1-j_1}\left((u_{j_1-2}(1-a) - b) + \frac{b}{a^{(2m+1)(2t+1)+2w+2}}\right)}, \qquad (3.38)$$

$$\widehat{r}_{2m+1}^{2w+1,2h+1} = \frac{(1+a)\left((u_{-1-j_1}(1-a) - b) + \frac{b}{a^{(2m+1)(2t+1)+2h+1}}\right)}{(1-a)a^{t+1-j_1}\left((v_{j_1-2}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2w+2} + \frac{\beta}{a^{(2m+1)(2t+1)+2w+2}}\right)}, \qquad (3.39)$$

where $s = 2w + 1$ and $s - t + j_1 = 2h + 1$, for $w, h \in \mathbb{N}_0$. From (3.38)–(3.39), we get

$$\lim_{m \to \infty} \widehat{p}_{2m+1}^{2w+1,2h+1} = \frac{(1-a)\left(v_{-1-j_1}(1+a) - \beta\right)}{(1+a)a^{t+1-j_1}\left(u_{j_1-2}(1-a) - b\right)}, \qquad (3.40)$$

$$\lim_{m \to \infty} \widehat{r}_{2m+1}^{2w+1,2h+1} = \frac{(1+a)\left(u_{-1-j_1}(1-a) - b\right)}{(a-1)a^{t+1-j_1}\left(v_{j_1-2}(1+a) - \beta\right)}, \qquad (3.41)$$

- s **is odd and** $s - t + j_1$ **is even:** In this case we get

$$\widehat{p}_{2m+1}^{2w+1,2h} = \frac{(1-a)\left((v_{-1-j_1}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2h} + \frac{\beta}{a^{(2m+1)(2t+1)+2h}}\right)}{(1+a)a^{t+1-j_1}\left((u_{j_1-2}(1-a) - b) + \frac{b}{a^{(2m+1)(2t+1)+2w+2}}\right)}, \qquad (3.42)$$

$$\widehat{r}_{2m+1}^{2w+1,2h} = \frac{(1+a)\left((u_{-1-j_1}(1-a) - b) + \frac{b}{a^{(2m+1)(2t+1)+2h}}\right)}{(1-a)a^{t+1-j_1}\left((v_{j_1-2}(1+a) - \beta)(-1)^{(2m+1)(2t+1)+2w+2} + \frac{\beta}{a^{(2m+1)(2t+1)+2w+2}}\right)}, \qquad (3.43)$$

where $s = 2w + 1$ and $s - t + j_1 = 2h$, for $w, h \in \mathbb{N}_0$. From (3.42)–(3.43), we have

$$\lim_{m \to \infty} \hat{p}_{2m+1}^{2w+1,2h} = \frac{(a-1)\,(v_{-1-j_1}(1+a) - \beta)}{(1+a)a^{t+1-j_1}\,(u_{j_1-2}(1-a) - b)}, \tag{3.44}$$

$$\lim_{m \to \infty} \hat{r}_{2m+1}^{2w+1,2h} = \frac{(1+a)\,(u_{-1-j_1}(1-a) - b)}{(a-1)a^{t+1-j_1}\,(v_{j_1-2}(1+a) - \beta)}. \tag{3.45}$$

From (3.32), (3.33), (3.36), (3.37), (3.40), (3.41), (3.44), and (3.45), the results can be seen easily. $\qquad\square$

3.2. Case $a \neq 1$, $\alpha = 1$

Here we introduce the asymptotic behavior of the solution of system (3.1) for the case $a \neq 1$ and $\alpha = 1$.

Theorem 3.4 *Suppose that $k \in \mathbb{N}$, $a \neq 1$, $\alpha = 1$, $b \neq 0 \neq \beta$, and $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements are true.*

(a) *If $|a| > 1$ and $u_{j_1-2} \neq b/(1-a) \neq u_{-1-j_1}$, for some $j_1 \in \{0,1\}$, then $x_{2(2t+1)m+2s+j_1} \to 0$ and $|y_{2(2t+1)m+2s+j_1}| \to \infty$, as $m \to \infty$.*

(b) *If $|a| < 1$ or $a = -1$ and $u_{j_1-2} = b/(1-a) = u_{-1-j_1}$, for some $j_1 \in \{0,1\}$, then $y_{2(2t+1)m+2s+j_1} \to 0$ and $|x_{2(2t+1)m+2s+j_1}| \to \infty$, as $m \to \infty$.*

Proof Let

$$\hat{p}_m^{t,2s+j_1} = \frac{(1-a)\,(v_{-1-j_1} + \beta\,(m\,(2t+1) + s - t + j_1))}{b + a^{m(2t+1)+s+1}\,(u_{j_1-2}\,(1-a) - b)}$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, $j_1 \in \{0,1\}$,

$$\hat{r}_m^{t,2s+j_1} = \frac{b + a^{m(2t+1)+s-t+j_1}\,(u_{-1-j_1}\,(1-a) - b)}{(1-a)\,(v_{j_1-2} + \beta\,(m\,(2t+1) + s + 1))}$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, $j_1 \in \{0,1\}$.
(a) Note that in this case

$$\lim_{m \to \infty} \hat{p}_m^{t,2s+j_1} = 0, \quad \lim_{m \to \infty} |\hat{r}_m^{t,2s+j_1}| = \infty,$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0,1\}$, from which the result can be obtained easily.
(b) In this case

$$\lim_{m \to \infty} |\hat{p}_m^{t,2s+j_1}| = \infty, \quad \lim_{m \to \infty} \hat{r}_m^{t,2s+j_1} = 0,$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0,1\}$, from which the result can be seen easily.

$\qquad\square$

3.3. Case $a = 1$, $\alpha \neq 1$

Here we study the asymptotic behavior of the solution of system (3.1) for the case $a = 1$ and $\alpha \neq 1$.

Theorem 3.5 *Suppose that* $k \in \mathbb{N}$, $a = 1$, $\alpha \neq 1$, $b \neq 0 \neq \beta$, *and* $(x_n, y_n)_{n \geq -k-2}$ *is a well-defined solution of system (3.1). Then the next statements hold.*

(a) *If* $|\alpha| > 1$ *and* $v_{j_1-2} \neq \beta/(1-\alpha) \neq u_{-1-j_1}$, *for some* $j_1 \in \{0,1\}$, *then* $y_{2(2t+1)m+2s+j_1} \to 0$ *and* $|x_{2(2t+1)m+2s+j_1}| \to \infty$, *as* $m \to \infty$.

(b) *If* $|\alpha| < 1$ *or* $\alpha = -1$ *and* $v_{j_1-2} = \beta/(1-\alpha) = v_{-1-j_1}$, *for some* $j_1 \in \{0,1\}$, *then* $x_{2(2t+1)m+2s+j_1} \to 0$ *and* $|y_{2(2t+1)m+2s+j_1}| \to \infty$, *as* $m \to \infty$.

Proof Let

$$\widetilde{p}_m^{t,2s+j_1} = \frac{\beta + \alpha^{m(2t+1)+s-t+j_1}\left(v_{-1-j_1}\left(1-\alpha\right) - \beta\right)}{(1-\alpha)\left(u_{j_1-2} + b\left(m\left(2t+1\right)+s+1\right)\right)}$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, $j_1 \in \{0,1\}$,

$$\widetilde{r}_m^{t,2s+j_1} = \frac{(1-\alpha)\left(u_{-1-j_1} + b\left(m\left(2t+1\right)+s-t+j_1\right)\right)}{\beta + \alpha^{m(2t+1)+s+1}\left(v_{j_1-2}\left(1-\alpha\right) - \beta\right)}$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, $j_1 \in \{0,1\}$.

(a): Note that in this case

$$\lim_{m\to\infty} |\widetilde{p}_m^{t,2s+j_1}| = \infty, \quad \lim_{m\to\infty} \widetilde{r}_m^{t,2s+j_1} = 0,$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0,1\}$, from which the result can be seen easily.

(b): In this case

$$\lim_{m\to\infty} \widetilde{p}_m^{t,2s+j_1} = 0, \quad \lim_{m\to\infty} |\widetilde{r}_m^{t,2s+j_1}| = \infty,$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0,1\}$, from which the result can be seen easily. $\qquad\square$

3.4. Case $a = 1$, $\alpha = 1$

Here we describe the asymptotic behavior of solution to system (3.1) for the case $a = 1$ and $\alpha = 1$.

Theorem 3.6 *Suppose that* $k \in \mathbb{N}$, $a = 1 = \alpha$, $b \neq 0 \neq \beta$, *and* $(x_n, y_n)_{n \geq -k-2}$ *is a well-defined solution of system (3.1). Then the next statements hold.*

(a) *If* $|\beta| < |b|$, *then* $x_n \to 0$ *and* $|y_n| \to \infty$, *as* $n \to \infty$.

(b) *If* $|\beta| > |b|$, *then* $y_n \to 0$ *and* $|x_n| \to \infty$, *as* $n \to \infty$.

(c) *If* $\beta = b$, $(t - j_1 + 1) \neq 0$, *and* $\frac{1}{t-j_1+1}\left(v_{-1-j_1} - u_{j_1-2}\right) = b$, *then the sequence* $x_{2(2t+1)m+2s+j_1}$, *for* $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, $j_1 \in \{0,1\}$, *is convergent.*

(d) If $\beta = b$ and $\frac{1}{b}\left(v_{-1-j_1} - u_{j_1-2}\right) > t - j_1 + 1$, then $|x_n| \to \infty$, as $n \to \infty$.

(e) If $\beta = b$ and $\frac{1}{b}\left(v_{-1-j_1} - u_{j_1-2}\right) < t - j_1 + 1$, then $x_n \to 0$, as $n \to \infty$.

(f) If $\beta = b$, $(t - j_1 + 1) \neq 0$ and $\frac{1}{t-j_1+1}\left(u_{-1-j_1} - v_{j_1-2}\right) = b$, then the sequence $y_{2(2t+1)m+2s+j_1}$, for $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, $j_1 \in \{0,1\}$, is convergent.

(g) If $\beta = b$ and $\frac{1}{b}\left(u_{-1-j_1} - v_{j_1-2}\right) > t - j_1 + 1$, then $|y_n| \to \infty$, as $n \to \infty$.

(h) If $\beta = b$ and $\frac{1}{b}\left(u_{-1-j_1} - v_{j_1-2}\right) < t - j_1 + 1$, then $y_n \to 0$, as $n \to \infty$.

(i) If $b = -\beta$, $(t - j_1 + 1) \neq 0$ and $b = -\frac{1}{t-j_1+1}\left(v_{-1-j_1} + u_{j_1-2}\right)$, then the sequences $x_{4(2t+1)m+2s+j_1}$, $x_{4(2t+1)m+4t+2+2s+j_1}$, for $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, $j_1 \in \{0,1\}$, are convergent.

(j) If $b = -\beta$ and $\frac{1}{b}\left(v_{-1-j_1} + u_{j_1-2}\right) + t - j_1 + 1 < 0$, then $|x_n| \to \infty$, as $n \to \infty$.

(k) If $b = -\beta$ and $\frac{1}{b}\left(v_{-1-j_1} + u_{j_1-2}\right) + t - j_1 + 1 > 0$, then $x_n \to 0$, as $n \to \infty$.

(l) If $b = -\beta$, $(t - j_1 + 1) \neq 0$ and $b = \frac{1}{t-j_1+1}\left(u_{-1-j_1} + v_{j_1-2}\right)$, then the sequences $y_{4(2t+1)m+2s+j_1}$, $y_{4(2t+1)m+4t+2+2s+j_1}$, for $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, $j_1 \in \{0,1\}$, are convergent.

(m) If $b = -\beta$ and $\frac{1}{b}\left(u_{-1-j_1} + v_{j_1-2}\right) > t - j_1 + 1$, then $|y_n| \to \infty$, as $n \to \infty$.

(n) If $b = -\beta$ and $\frac{1}{b}\left(u_{-1-j_1} + v_{j_1-2}\right) < t - j_1 + 1$, then $y_n \to 0$, as $n \to \infty$.

Proof (a)–(b): From Eq. (3.8) and Eq. (3.9) we get

$$
\begin{aligned}
x_{2(2t+1)m+2s+j_1} &= x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{v_{-1-j_1} + \beta\left(p(2t+1) + s - t + j_1\right)}{u_{j_1-2} + b\left(p(2t+1) + s + 1\right)} \\
&= x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{\beta\left(1 + \frac{s-t+j_1+\frac{v_{-1-j_1}}{\beta}}{p(2t+1)}\right)}{b\left(1 + \frac{s+1+\frac{u_{j_1-2}}{b}}{p(2t+1)}\right)},
\end{aligned}
\tag{3.46}
$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0,1\}$,

$$
\begin{aligned}
y_{2(2t+1)m+2s+j_1} &= y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{u_{-1-j_1} + b\left(p(2t+1) + s - t + j_1\right)}{v_{j_1-2} + \beta\left(p(2t+1) + s + 1\right)} \\
&= y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{b\left(1 + \frac{s-t+j_1+\frac{u_{-1-j_1}}{b}}{p(2t+1)}\right)}{\beta\left(1 + \frac{s+1+\frac{v_{j_1-2}}{\beta}}{p(2t+1)}\right)},
\end{aligned}
\tag{3.47}
$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0,1\}$. From (3.46) and (3.47) the results can be seen easily.

(c)–(h): If $b = \beta$, from (3.46) and (3.47) and by employing the Taylor expansion for $(1 + x)^{-1}$ on the interval

$(-\epsilon, \epsilon)$, where $\epsilon > 0$, we get, for sufficiently large m,

$$x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{\left(1 + \frac{s-t+j_1+\frac{v_{-1-j_1}}{b}}{p(2t+1)}\right)}{\left(1 + \frac{s+1+\frac{u_{j_1-2}}{b}}{p(2t+1)}\right)}$$

$$= x_{2s+j_1-2(2t+1)} C_1(m_0) \prod_{p=m_0+1}^{m} \left(1 + \frac{\frac{v_{-1-j_1}-u_{j_1-2}}{b} - t + j_1 - 1}{p(2t+1)} + \mathcal{O}\left(\frac{1}{p^2}\right)\right)$$

$$= x_{2s+j_1-2(2t+1)} C_1(m_0) exp\left(\sum_{p=m_0+1}^{m} \left(\frac{\frac{v_{-1-j_1}-u_{j_1-2}}{b} - t + j_1 - 1}{p(2t+1)} + \mathcal{O}\left(\frac{1}{p^2}\right)\right)\right), (3.48)$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0, 1\}$,

$$y_{2(2t+1)m+2s+j_1} = y_{2s+j_1-2(2t+1)} \prod_{p=0}^{m} \frac{\left(1 + \frac{s-t+j_1+\frac{u_{-1-j_1}}{b}}{p(2t+1)}\right)}{\left(1 + \frac{s+1+\frac{v_{j_1-2}}{b}}{p(2t+1)}\right)}$$

$$= y_{2s+j_1-2(2t+1)} C_2(m_0) \prod_{p=m_0+1}^{m} \left(1 + \frac{\frac{u_{-1-j_1}-v_{j_1-2}}{b} - t + j_1 - 1}{p(2t+1)} + \mathcal{O}\left(\frac{1}{p^2}\right)\right)$$

$$= y_{2s+j_1-2(2t+1)} C_2(m_0) exp\left(\sum_{p=m_0+1}^{m} \left(\frac{\frac{u_{-1-j_1}-v_{j_1-2}}{b} - t + j_1 - 1}{p(2t+1)} + \mathcal{O}\left(\frac{1}{p^2}\right)\right)\right), (3.49)$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0, 1\}$, where $C_1(m_0) = \prod_{p=0}^{m_0} \frac{\left(1 + \frac{s-t+j_1+\frac{v_{-1-j_1}}{b}}{p(2t+1)}\right)}{\left(1 + \frac{s+1+\frac{u_{j_1-2}}{b}}{p(2t+1)}\right)}$ and

$C_2(m_0) = \prod_{p=0}^{m_0} \frac{\left(1 + \frac{s-t+j_1+\frac{u_{-1-j_1}}{b}}{p(2t+1)}\right)}{\left(1 + \frac{s+1+\frac{v_{j_1-2}}{b}}{p(2t+1)}\right)}$. From (3.48) and (3.49), the fact that $\sum_{p=m_0+1}^{m} \frac{1}{p} \to +\infty$, as $m \to \infty$,

and the convergence of the series $\sum_{p=m_0+1}^{+\infty} \mathcal{O}\left(\frac{1}{p^2}\right)$, the statements can be seen easily.

(i)-(n): If $-\beta = b$, from (3.46) and (3.47) by using the Taylor expansion for $(1+x)^{-1}$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, we get, for sufficiently large m,

$$x_{2(2t+1)m+2s+j_1} = x_{2s+j_1-2(2t+1)}(-1)^{m+1} \prod_{p=0}^{m} \frac{\left(1 + \frac{s-t+j_1-\frac{v_{-1-j_1}}{b}}{p(2t+1)}\right)}{\left(1 + \frac{s+1+\frac{u_{j_1-2}}{b}}{p(2t+1)}\right)}$$

$$= x_{2s+j_1-2(2t+1)}(-1)^{m+1} C_1(m_1) \prod_{p=m_1+1}^{m} \left(1 - \frac{\frac{v_{-1-j_1}+u_{j_1-2}}{b} + t - j_1 + 1}{p(2t+1)} + \mathcal{O}\left(\frac{1}{p^2}\right)\right)$$

$$= x_{2s+j_1-2(2t+1)}(-1)^{m+1} C_1(m_1) exp\left(-\sum_{p=m_1+1}^{m} \left(\frac{\frac{v_{-1-j_1}+u_{j_1-2}}{b} + t - j_1 + 1}{p(2t+1)} + \mathcal{O}\left(\frac{1}{p^2}\right)\right)\right), (3.50)$$

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\},$

$$
\begin{aligned}
y_{2(2t+1)m+2s+j_1} &= y_{2s+j_1-2(2t+1)}(-1)^{m+1}\prod_{p=0}^{m}\frac{\left(1 + \frac{s-t+j_1+\frac{u-1-j_1}{b}}{p(2t+1)}\right)}{\left(1 + \frac{s+1-\frac{v_{j_1}-2}{b}}{p(2t+1)}\right)} \\
&= y_{2s+j_1-2(2t+1)}(-1)^{m+1}C_2(m_1)\prod_{p=m_1+1}^{m}\left(1 + \frac{\frac{u-1-j_1+v_{j_1}-2}{b} - t + j_1 - 1}{p(2t+1)} + \mathcal{O}\left(\frac{1}{p^2}\right)\right) \\
&= y_{2s+j_1-2(2t+1)}(-1)^{m+1}C_2(m_1)exp\left(\sum_{p=m_1+1}^{m}\left(\frac{\frac{u-1-j_1+v_{j_1}-2}{b} - t + j_1 - 1}{p(2t+1)} + \mathcal{O}\left(\frac{1}{p^2}\right)\right)\right), (3.51)
\end{aligned}
$$

for each $2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}, j_1 \in \{0, 1\}$, where $C_1(m_1) = \prod_{p=0}^{m_1}\frac{\left(1 + \frac{s-t+j_1-\frac{v-1-j_1}{b}}{p(2t+1)}\right)}{\left(1 + \frac{s+1+\frac{u_{j_1}-2}{b}}{p(2t+1)}\right)}$ and

$C_2(m_1) = \prod_{p=0}^{m_1}\frac{\left(1 + \frac{s-t+j_1+\frac{u-1-j_1}{b}}{p(2t+1)}\right)}{\left(1 + \frac{s+1-\frac{v_{j_1}-2}{b}}{p(2t+1)}\right)}$. From (3.50) and (3.51), the fact that $\sum_{p=m_1+1}^{m}\frac{1}{p} \to +\infty$, as $m \to \infty$,

and the convergence of the series $\sum_{p=m_1+1}^{+\infty}\mathcal{O}\left(\frac{1}{p^2}\right)$, the statements in (i)–(n) can be obtained easily. $\qquad\square$

3.5. Case $a = -1, |\alpha| < 1$

Here we introduce the asymptotic behavior of well-defined solutions of system (3.1) for the case $a = -1$ and $|\alpha| < 1$, by using the next two notations:

$$
\widetilde{K}_{j_1} := \frac{2\beta}{(1 - \alpha)\left(b + (-1)^{s+1}\left(2u_{j_1-2} - b\right)\right)}, \quad \widetilde{L}_{j_1} := \frac{(1 - \alpha)\left(b + (-1)^{s-t+j_1}\left(2u_{-1-j_1} - b\right)\right)}{2\beta},
$$

where $m, t \in \mathbb{N}_0, \ j_1 \in \{0, 1\}$.

Theorem 3.7 *Suppose that $a = -1, |\alpha| < 1, b \neq 0 \neq \beta, m$ is even, and $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements are true.*

(a) *If $|\widetilde{K}_{j_1}| < 1$ for some $j_1 \in \{0, 1\}, 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then $x_{4(2t+1)m+2s+j_1} \to 0$ as $m \to \infty$.*

(b) *If $|\widetilde{K}_{j_1}| > 1$ for some $j_1 \in \{0, 1\}, 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then $|x_{4(2t+1)m+2s+j_1}| \to \infty$ as $m \to \infty$.*

(c) *If $\widetilde{K}_{j_1} = 1$ for some $j_1 \in \{0, 1\}, 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then the sequence $x_{4(2t+1)m+2s+j_1}$ is convergent.*

(d) *If $\widetilde{K}_{j_1} = -1$ for some $j_1 \in \{0, 1\}, 2s + j_1 \in \{k - 2, k - 1, \ldots, 3k - 3\}$, then the sequences $x_{8(2t+1)m+2s+j_1}$, $x_{8(2t+1)m+8t+4+2s+j_1}$ are convergent.*

(e) If $|\widetilde{L}_{j_1}| < 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $y_{4(2t+1)m+2s+j_1} \to 0$ as $m \to \infty$.

(f) If $|\widetilde{L}_{j_1}| > 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $|y_{4(2t+1)m+2s+j_1}| \to \infty$ as $m \to \infty$.

(g) If $\widetilde{L}_{j_1} = 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequence $y_{4(2t+1)m+2s+j_1}$ is convergent.

(h) If $\widetilde{L}_{j_1} = -1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequences $y_{8(2t+1)m+2s+j_1}$, $y_{8(2t+1)m+8t+4+2s+j_1}$ are convergent.

Proof (a),(b) and (e),(f): By using (3.12) and (3.13) for the case $a = -1$ and $|\alpha| < 1$,

$$\lim_{m \to \infty} \frac{2\left(\beta + \alpha^{2m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-\alpha) - \beta\right)\right)}{(1-\alpha)\left(b + (-1)^{2m(2t+1)+s+1}(2u_{j_1-2} - b)\right)} = \widetilde{K}_{j_1}, \tag{3.52}$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0,1\}$,

$$\lim_{m \to \infty} \frac{(1-\alpha)\left(b + (-1)^{2m(2t+1)+s-t+j_1}(2u_{-1-j_1} - b)\right)}{2\left(\beta + \alpha^{2m(2t+1)+s+1}(v_{j_1-2}(1-\alpha) - \beta)\right)} = \widetilde{L}_{j_1}, \tag{3.53}$$

for each $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0,1\}$, from which, along with formulas (3.52) and (3.53), the results in (a), (b), (e), and (f) can be seen easily.

(c): From (3.12), we get that

$$\frac{2\left(\beta + \alpha^{2m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-\alpha) - \beta\right)\right)}{(1-\alpha)\left(b + (-1)^{2m(2t+1)+s+1}(2u_{j_1-2} - b)\right)} = 1 + \frac{\alpha^{2m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-\alpha) - \beta\right)}{\beta},$$

which holds for $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$ and $j_1 \in \{0,1\}$. By using Taylor expansion for $\ln(1+x)$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, in the above equation, the statement easily follows.

(d): The result can be easily obtained from the relation

$$\frac{2\left(\beta + \alpha^{2m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-\alpha) - \beta\right)\right)}{(1-\alpha)\left(b + (-1)^{2m(2t+1)+s+1}(2u_{j_1-2} - b)\right)} = -\left(1 + \frac{\alpha^{2m(2t+1)+s-t+j_1}\left(v_{-1-j_1}(1-\alpha) - \beta\right)}{\beta}\right),$$

which holds for $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}, j_1 \in \{0,1\}$ and Taylor expansion for $\ln(1+x)$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$.

(g): By using (3.13) for the case $a = -1$ and $|\alpha| < 1$, we can write

$$\frac{(1-\alpha)\left(b + (-1)^{2m(2t+1)+s-t+j_1}(2u_{-1-j_1} - b)\right)}{2\left(\beta + \alpha^{2m(2t+1)+s+1}(v_{j_1-2}(1-\alpha) - \beta)\right)} = 1 - \frac{\alpha^{2m(2t+1)+s+1}(v_{j_1-2}(1-\alpha) - \beta)(1 + \mathcal{O}(1))}{\beta},$$

which holds for $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$ and $j_1 \in \{0,1\}$. By using Taylor expansion for $(1+x)^{-1}$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, in the above equation, the result can be seen easily.

(h): In this case by using (3.13) for the case $a = -1$ and $|\alpha| < 1$, we have

$$\frac{(1-\alpha)\left(b + (-1)^{2m(2t+1)+s-t+j_1}(2u_{-1-j_1} - b)\right)}{2\left(\beta + \alpha^{2m(2t+1)+s+1}(v_{j_1-2}(1-\alpha) - \beta)\right)} = -\left(1 - \frac{\alpha^{2m(2t+1)+s+1}(v_{j_1-2}(1-\alpha) - \beta)(1 + \mathcal{O}(1))}{\beta}\right),$$

which holds for $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$ and $j_1 \in \{0,1\}$, from which, along with Taylor expansion for $(1+x)^{-1}$ on the interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$, the statement follows. $\qquad\square$

Theorem 3.8 *Suppose that* $a = -1$, $|\alpha| < 1$, $b \neq 0 \neq \beta$, $m, t, w, h \in \mathbb{N}_0$, $j_1 \in \{0,1\}$ m *is odd, and* $(x_n, y_n)_{n \geq -k-2}$ *is a well-defined solution of system* (3.1). *Then the next statements are true.*

(a) *If* $\left| \frac{\beta}{(1-\alpha)u_{j_1-2}} \right| < 1$ *for some* $j_1 \in \{0,1\}$, *then* $x_{4(2t+1)m+4t+2+4w+j_1} \to 0$ *as* $m \to \infty$.

(b) *If* $\left| \frac{\beta}{(1-\alpha)u_{j_1-2}} \right| > 1$ *for some* $j_1 \in \{0,1\}$, *then* $\left| x_{4(2t+1)m+4t+2+4w+j_1} \right| \to \infty$ *as* $m \to \infty$.

(c) *If* $\frac{\beta}{(1-\alpha)u_{j_1-2}} = 1$ *for some* $j_1 \in \{0,1\}$, *then the sequence* $x_{4(2t+1)m+4t+2+4w+j_1}$ *is convergent.*

(d) *If* $\frac{\beta}{(1-\alpha)u_{j_1-2}} = -1$ *for some* $j_1 \in \{0,1\}$, *then the sequences* $x_{8(2t+1)m+4t+2+4w+j_1}$, $x_{8(2t+1)m+12t+6+4w+j_1}$
are convergent.

(e) *If* $\left| \frac{(1-\alpha)u_{-1-j_1}}{\beta} \right| < 1$ *for some* $j_1 \in \{0,1\}$, *then* $y_{4(2t+1)m+4t+2+4w+j_1} \to 0$, $y_{4(2t+1)m+4t+4w+4+j_1} \to 0$ *as* $m \to \infty$.

(f) *If* $\left| \frac{(1-\alpha)u_{-1-j_1}}{\beta} \right| > 1$ *for some* $j_1 \in \{0,1\}$, *then* $\left| y_{4(2t+1)m+4t+2+4w+j_1} \right| \to \infty$, $\left| y_{4(2t+1)m+4t+4w+4+j_1} \right| \to \infty$ *as* $m \to \infty$.

(g) *If* $\frac{(1-\alpha)u_{-1-j_1}}{\beta} = 1$ *for some* $j_1 \in \{0,1\}$, *then the sequences* $y_{4(2t+1)m+4t+2+4w+j_1}$, $y_{4(2t+1)m+4t+4w+4+j_1}$ *are convergent.*

(h) *If* $\frac{(1-\alpha)u_{-1-j_1}}{\beta} = -1$ *for some* $j_1 \in \{0,1\}$, *then the sequences* $y_{8(2t+1)m+4t+2+4w+j_1}$, $y_{8(2t+1)m+12t+6+4w+j_1}$, $y_{8(2t+1)m+4t+4w+4+j_1}$, $y_{8(2t+1)m+12t+4w+8+j_1}$ *are convergent.*

(i) *If* $\left| \frac{\beta}{(1-\alpha)\left(b-u_{j_1-2}\right)} \right| < 1$ *for some* $j_1 \in \{0,1\}$, *then* $x_{4(2t+1)m+4t+4w+4+j_1} \to 0$ *as* $m \to \infty$.

(j) *If* $\left| \frac{\beta}{(1-\alpha)\left(b-u_{j_1-2}\right)} \right| > 1$ *for some* $j_1 \in \{0,1\}$, *then* $\left| x_{4(2t+1)m+4t+4w+4+j_1} \right| \to \infty$ *as* $m \to \infty$.

(k) *If* $\frac{\beta}{(1-\alpha)\left(b-u_{j_1-2}\right)} = 1$ *for some* $j_1 \in \{0,1\}$, *then the sequence* $x_{4(2t+1)m+4t+4w+4+j_1}$ *is convergent.*

(l) *If* $\frac{\beta}{(1-\alpha)\left(b-u_{j_1-2}\right)} = -1$ *for some* $j_1 \in \{0,1\}$, *then the sequences* $x_{8(2t+1)m+4t+4w+4+j_1}$, $x_{8(2t+1)m+12t+4w+8+j_1}$
are convergent.

(m) *If* $\left| \frac{(1-\alpha)\left(b-u_{-1-j_1}\right)}{\beta} \right| < 1$ *for some* $j_1 \in \{0,1\}$, *then* $y_{4(2t+1)m+4t+2+4w+j_1} \to 0$, $y_{4(2t+1)m+4t+4w+4+j_1} \to 0$ *as* $m \to \infty$.

(n) *If* $\left| \frac{(1-\alpha)\left(b-u_{-1-j_1}\right)}{\beta} \right| > 1$ *for some* $j_1 \in \{0,1\}$, *then* $\left| y_{4(2t+1)m+4t+2+4w+j_1} \right| \to \infty$, $\left| y_{4(2t+1)m+4t+4w+4+j_1} \right| \to \infty$ *as* $m \to \infty$.

(o) *If* $\frac{(1-\alpha)\left(b-u_{-1-j_1}\right)}{\beta} = 1$ *for some* $j_1 \in \{0,1\}$, *then the sequences* $y_{4(2t+1)m+4t+2+4w+j_1}$, $y_{4(2t+1)m+4t+4w+4+j_1}$ *are convergent.*

(p) If $\frac{(1-\alpha)\left(b-u_{-1-j_1}\right)}{\beta} = -1$ *for some* $j_1 \in \{0,1\}$, *then the sequences* $y_{8(2t+1)m+4t+2+4w+j_1}$, $y_{8(2t+1)m+12t+6+4w+j_1}$, $y_{8(2t+1)m+4t+4w+4+j_1}$, $y_{8(2t+1)m+12t+4w+8+j_1}$ *are convergent.*

Proof Let $a = -1$, $|\alpha| < 1$, and m be odd. From (3.12)-(3.13) we have

$$p_{2m+1}^{s,s-t+j_1} = \frac{2\left((v_{-1-j_1}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+s-t+j_1}+\beta\right)}{(1-\alpha)\left((2u_{j_1-2}-b)(-1)^{(2m+1)(2t+1)+s+1}+b\right)}, \tag{3.54}$$

$$r_{2m+1}^{s,s-t+j_1} = \frac{(1-\alpha)\left((2u_{-1-j_1}-b)(-1)^{(2m+1)(2t+1)+s-t+j_1}+b\right)}{2\left((v_{j_1-2}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+s+1}+\beta\right)}. \tag{3.55}$$

Here there are four cases to be considered.

- s **is even and** $s-t+j_1$ **is odd:** In this case we have

$$p_{2m+1}^{2w,2h+1} = \frac{2\left((v_{-1-j_1}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2h+1}+\beta\right)}{(1-\alpha)\left((2u_{j_1-2}-b)(-1)^{(2m+1)(2t+1)+2w+1}+b\right)}, \tag{3.56}$$

$$r_{2m+1}^{2w,2h+1} = \frac{(1-\alpha)\left((2u_{-1-j_1}-b)(-1)^{(2m+1)(2t+1)+2h+1}+b\right)}{2\left((v_{j_1-2}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2w+1}+\beta\right)}, \tag{3.57}$$

where $s = 2w$ and $s-t+j_1 = 2h+1$, for $w, h \in \mathbb{N}_0$. Taking the limit of both sides in equations (3.56) and (3.57), we get

$$\lim_{m\to\infty} p_{2m+1}^{2w,2h+1} = \lim_{m\to\infty} \frac{2\left((v_{-1-j_1}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2h+1}+\beta\right)}{(1-\alpha)2u_{j_1-2}} = \frac{\beta}{(1-\alpha)u_{j_1-2}}, \tag{3.58}$$

$$\lim_{m\to\infty} r_{2m+1}^{2w,2h+1} = \lim_{m\to\infty} \frac{(1-\alpha)2u_{-1-j_1}}{2\left((v_{j_1-2}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2w+1}+\beta\right)} = \frac{(1-\alpha)u_{-1-j_1}}{\beta}. \tag{3.59}$$

- s **and** $s-t+j_1$ **are both even:** In this case we get

$$p_{2m+1}^{2w,2h} = \frac{2\left((v_{-1-j_1}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2h}+\beta\right)}{(1-\alpha)\left((2u_{j_1-2}-b)(-1)^{(2m+1)(2t+1)+2w+1}+b\right)}, \tag{3.60}$$

$$r_{2m+1}^{2w,2h} = \frac{(1-\alpha)\left((2u_{-1-j_1}-b)(-1)^{(2m+1)(2t+1)+2h}+b\right)}{2\left((v_{j_1-2}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2w+1}+\beta\right)}, \tag{3.61}$$

where $s = 2w$ and $s-t+j_1 = 2h$, for $w, h \in \mathbb{N}_0$. Taking the limit of both sides in equations (3.60) and (3.61), we have

$$\lim_{m\to\infty} p_{2m+1}^{2w,2h} = \lim_{m\to\infty} \frac{2\left((v_{-1-j_1}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2h}+\beta\right)}{(1-\alpha)2u_{j_1-2}} = \frac{\beta}{(1-\alpha)u_{j_1-2}}, \tag{3.62}$$

$$\lim_{m\to\infty} r_{2m+1}^{2w,2h} = \lim_{m\to\infty} \frac{(1-\alpha)\left(2b-2u_{-1-j_1}\right)}{2\left((v_{j_1-2}(1-\alpha)-\beta)\alpha^{(2m+1)(2t+1)+2w+1}+\beta\right)} = \frac{(1-\alpha)\left(b-u_{-1-j_1}\right)}{\beta}. \tag{3.63}$$

- **s and $s - t + j_1$ are both odd:** In this case we have

$$p_{2m+1}^{2w+1,2h+1} = \frac{2\left((v_{-1-j_1}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2h+1} + \beta\right)}{(1-\alpha)\left((2u_{j_1-2} - b)(-1)^{(2m+1)(2t+1)+2w+2} + b\right)}, \tag{3.64}$$

$$r_{2m+1}^{2w+1,2h+1} = \frac{(1-\alpha)\left((2u_{-1-j_1} - b)(-1)^{(2m+1)(2t+1)+2h+1} + b\right)}{2\left((v_{j_1-2}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2w+2} + \beta\right)}, \tag{3.65}$$

where $s = 2w + 1$ and $s - t + j_1 = 2h + 1$, for $w, h \in \mathbb{N}_0$. Taking the limit of both sides in equations (3.64) and (3.65), we obtain

$$\lim_{m\to\infty} p_{2m+1}^{2w+1,2h+1} = \lim_{m\to\infty} \frac{2\left((v_{-1-j_1}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2h+1} + \beta\right)}{(1-\alpha)\left(2b - 2u_{j_1-2}\right)} = \frac{\beta}{(1-\alpha)\left(b - u_{j_1-2}\right)}, \tag{3.66}$$

$$\lim_{m\to\infty} r_{2m+1}^{2w+1,2h+1} = \lim_{m\to\infty} \frac{(1-\alpha)2u_{-1-j_1}}{2\left((v_{j_1-2}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2w+2} + \beta\right)} = \frac{(1-\alpha)u_{-1-j_1}}{\beta}. \tag{3.67}$$

- **s is odd and $s - t + j_1$ is even:** In this case we get

$$p_{2m+1}^{2w+1,2h} = \frac{2\left((v_{-1-j_1}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2h} + \beta\right)}{(1-\alpha)\left((2u_{j_1-2} - b)(-1)^{(2m+1)(2t+1)+2w+2} + b\right)}, \tag{3.68}$$

$$r_{2m+1}^{2w+1,2h} = \frac{(1-\alpha)\left((2u_{-1-j_1} - b)(-1)^{(2m+1)(2t+1)+2h} + b\right)}{2\left((v_{j_1-2}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2w+2} + \beta\right)}, \tag{3.69}$$

where $s = 2w + 1$ and $s - t + j_1 = 2h$, for $w, h \in \mathbb{N}_0$. Similarly, taking the limit of both sides in equations (3.68) and (3.69), we have

$$\lim_{m\to\infty} p_{2m+1}^{2w+1,2h} = \lim_{m\to\infty} \frac{2\left((v_{-1-j_1}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2h} + \beta\right)}{(1-\alpha)\left(2b - 2u_{j_1-2}\right)} = \frac{\beta}{(1-\alpha)\left(b - u_{j_1-2}\right)}, \tag{3.70}$$

$$\lim_{m\to\infty} r_{2m+1}^{2w+1,2h} = \lim_{m\to\infty} \frac{(1-\alpha)\left(2b - 2u_{-1-j_1}\right)}{2\left((v_{j_1-2}(1-\alpha) - \beta)\alpha^{(2m+1)(2t+1)+2w+2} + \beta\right)} = \frac{(1-\alpha)\left(b - u_{-1-j_1}\right)}{\beta}. \tag{3.71}$$

From (3.58), (3.59), (3.62), (3.63), (3.66), (3.67), (3.70), and (3.71), the results of the theorem can be seen easily.

$\square$

3.6. Case $|a| < 1$, $\alpha = -1$

Here, by employing the next two notations,

$$M_{j_1} := \frac{(1-a)\left(\beta + (-1)^{s-t+j_1}\left(2v_{-1-j_1} - \beta\right)\right)}{2b}, \quad P_{j_1} := \frac{2b}{(1-a)\left(\beta + (-1)^{s+1}\left(2v_{j_1-2} - \beta\right)\right)},$$

where $m, t \in \mathbb{N}_0$, $j_1 \in \{0, 1\}$, for nonnegative even integer number m we can first give the following theorem, which can be proven like Theorem 3.7.

Theorem 3.9 *Suppose that $\alpha = -1$, $|a| < 1$, $b \neq 0 \neq \beta$, m is even, and $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements hold.*

(a) *If $|M_{j_1}| < 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $x_{4(2t+1)m+2s+j_1} \to 0$ as $m \to \infty$.*

(b) *If $|M_{j_1}| > 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $|x_{4(2t+1)m+2s+j_1}| \to \infty$ as $m \to \infty$.*

(c) *If $M_{j_1} = 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequence $x_{4(2t+1)m+2s+j_1}$ is convergent.*

(d) *If $M_{j_1} = -1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequences $x_{8(2t+1)m+2s+j_1}$, $x_{8(2t+1)m+8t+4+2s+j_1}$ are convergent.*

(e) *If $|P_{j_1}| < 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $y_{4(2t+1)m+2s+j_1} \to 0$ as $m \to \infty$.*

(f) *If $|P_{j_1}| > 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then $|y_{4(2t+1)m+2s+j_1}| \to \infty$ as $m \to \infty$.*

(g) *If $P_{j_1} = 1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequence $y_{4(2t+1)m+2s+j_1}$ is convergent.*

(h) *If $P_{j_1} = -1$ for some $j_1 \in \{0,1\}$, $2s + j_1 \in \{k-2, k-1, \ldots, 3k-3\}$, then the sequences $y_{8(2t+1)m+2s+j_1}$, $y_{8(2t+1)m+8t+4+2s+j_1}$ are convergent.*

Similarly, for nonnegative odd integer number m we can give the next theorem, which can be proven like Theorem 3.8.

Theorem 3.10 *Suppose that $\alpha = -1$, $|a| < 1$, $b \neq 0 \neq \beta$, $m, t, w \in \mathbb{N}_0$, $j_1 \in \{0,1\}$, m is odd, and $(x_n, y_n)_{n \geq -k-2}$ is a well-defined solution of system (3.1). Then the next statements hold.*

(a) *If $|\frac{b}{(1-a)v_{j_1-2}}| < 1$ for some $j_1 \in \{0,1\}$, then $y_{4(2t+1)m+4t+2+4w+j_1} \to 0$ as $m \to \infty$.*

(b) *If $|\frac{b}{(1-a)v_{j_1-2}}| > 1$ for some $j_1 \in \{0,1\}$, then $|y_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$ as $m \to \infty$.*

(c) *If $\frac{b}{(1-a)v_{j_1-2}} = 1$ for some $j_1 \in \{0,1\}$, then the sequence $y_{4(2t+1)m+4t+2+4w+j_1}$ is convergent.*

(d) *If $\frac{b}{(1-a)v_{j_1-2}} = -1$ for some $j_1 \in \{0,1\}$, then the sequences $y_{8(2t+1)m+4t+2+4w+j_1}$, $y_{8(2t+1)m+12t+6+4w+j_1}$ are convergent.*

(e) *If $|\frac{(1-a)v_{-1-j_1}}{b}| < 1$ for some $j_1 \in \{0,1\}$, then $x_{4(2t+1)m+4t+2+4w+j_1} \to 0$, $x_{4(2t+1)m+4t+4w+4+j_1} \to 0$ as $m \to \infty$.*

(f) *If $|\frac{(1-a)v_{-1-j_1}}{b}| > 1$ for some $j_1 \in \{0,1\}$, then $|x_{4(2t+1)m+4t+2+4w+j_1}| \to \infty$, $|x_{4(2t+1)m+4t+4w+4+j_1}| \to \infty$ as $m \to \infty$.*

(g) *If $\frac{(1-a)v_{-1-j_1}}{b} = 1$ for some $j_1 \in \{0,1\}$, then the sequences $x_{4(2t+1)m+4t+2+4w+j_1}$, $x_{4(2t+1)m+4t+4w+4+j_1}$ are convergent.*

(h) If $\frac{(1-a)v_{-1-j_1}}{b} = -1$ for some $j_1 \in \{0,1\}$, then the sequences $x_{8(2t+1)m+4t+2+4w+j_1}$, $x_{8(2t+1)m+12t+6+4w+j_1}$, $x_{8(2t+1)m+4t+4w+4+j_1}$, $x_{8(2t+1)m+12t+4w+8+j_1}$ are convergent.

(i) If $\left|\frac{b}{(1-a)\left(\beta-v_{j_1-2}\right)}\right| < 1$ for some $j_1 \in \{0,1\}$, then $y_{4(2t+1)m+4t+4w+4+j_1} \to 0$ as $m \to \infty$.

(j) If $\left|\frac{b}{(1-a)\left(\beta-v_{j_1-2}\right)}\right| > 1$ for some $j_1 \in \{0,1\}$, then $\left|y_{4(2t+1)m+4t+4w+4+j_1}\right| \to \infty$ as $m \to \infty$.

(k) If $\frac{b}{(1-a)\left(\beta-v_{j_1-2}\right)} = 1$ for some $j_1 \in \{0,1\}$, then the sequence $y_{4(2t+1)m+4t+4w+4+j_1}$ is convergent.

(l) If $\frac{b}{(1-a)\left(\beta-v_{j_1-2}\right)} = -1$ for some $j_1 \in \{0,1\}$, then the sequences $y_{8(2t+1)m+4t+4w+4+j_1}$, $y_{8(2t+1)m+12t+4w+8+j_1}$ are convergent.

(m) If $\left|\frac{(1-a)\left(\beta-v_{-1-j_1}\right)}{b}\right| < 1$ for some $j_1 \in \{0,1\}$, then $x_{4(2t+1)m+4t+2+4w+j_1} \to 0$, $x_{4(2t+1)m+4t+4w+4+j_1} \to 0$ as $m \to \infty$.

(n) If $\left|\frac{(1-a)\left(\beta-v_{-1-j_1}\right)}{b}\right| > 1$ for some $j_1 \in \{0,1\}$, then $\left|x_{4(2t+1)m+4t+2+4w+j_1}\right| \to \infty$, $\left|x_{4(2t+1)m+4t+4w+4+j_1}\right| \to \infty$ as $m \to \infty$.

(o) If $\frac{(1-a)\left(\beta-v_{-1-j_1}\right)}{b} = 1$ for some $j_1 \in \{0,1\}$, then the sequences $x_{4(2t+1)m+4t+2+4w+j_1}$, $x_{4(2t+1)m+4t+4w+4+j_1}$ are convergent.

(p) If $\frac{(1-a)\left(\beta-v_{-1-j_1}\right)}{b} = -1$ for some $j_1 \in \{0,1\}$, then the sequences $x_{8(2t+1)m+4t+2+4w+j_1}$, $x_{8(2t+1)m+12t+6+4w+j_1}$, $x_{8(2t+1)m+4t+4w+4+j_1}$, $x_{8(2t+1)m+12t+4w+8+j_1}$ are convergent.

Theorem 3.11 *Suppose that* $a = -1 = \alpha$, $b \neq 0 \neq \beta$, k *is odd, and* $(x_n, y_n)_{n \geq -k-2}$ *is a well-defined solution of system* (3.1). *Then the next statements hold.*

(a) If $\left|\frac{x_{-2}y_{-k-2}}{y_{-1}x_{-k-1}} \frac{1}{bx_{-2}y_{-k-2}-1}\right| < 1$, then $x_{4n} \to 0$, $|y_{4n+3}| \to \infty$ as $n \to \infty$.

(b) If $\left|\frac{x_{-2}y_{-k-2}}{y_{-1}x_{-k-1}} \frac{1}{bx_{-2}y_{-k-2}-1}\right| > 1$, then $|x_{4n}| \to \infty$, $y_{4n+3} \to 0$ as $n \to \infty$.

(c) If $\frac{x_{-2}y_{-k-2}}{y_{-1}x_{-k-1}} \frac{1}{bx_{-2}y_{-k-2}-1} = 1$, then x_{4n} and y_{4n+3} are $2k$-periodic.

(d) If $\frac{x_{-2}y_{-k-2}}{y_{-1}x_{-k-1}} \frac{1}{bx_{-2}y_{-k-2}-1} = -1$, then x_{4n} and y_{4n+3} are $4k$-periodic.

(e) If $\left|\frac{x_{-2}y_{-k-2}}{y_{-1}x_{-k-1}} \left(\beta y_{-1}x_{-k-1} - 1\right)\right| < 1$, then $x_{4n+2} \to 0$, $|y_{4n+1}| \to \infty$ as $n \to \infty$.

(f) If $\left|\frac{x_{-2}y_{-k-2}}{y_{-1}x_{-k-1}} \left(\beta y_{-1}x_{-k-1} - 1\right)\right| > 1$, then $|x_{4n+2}| \to \infty$, $y_{4n+1} \to 0$ as $n \to \infty$.

(g) If $\frac{x_{-2}y_{-k-2}}{y_{-1}x_{-k-1}} \left(\beta y_{-1}x_{-k-1} - 1\right) = 1$, then x_{4n+2} and y_{4n+1} are $2k$-periodic.

(h) If $\frac{x_{-2}y_{-k-2}}{y_{-1}x_{-k-1}} \left(\beta y_{-1}x_{-k-1} - 1\right) = -1$, then x_{4n+2} and y_{4n+1} are $4k$-periodic.

(i) If $\left|\frac{x_{-1}y_{-k-1}}{y_{-2}x_{-k-2}} \frac{1}{bx_{-1}y_{-k-1}-1}\right| < 1$, then $x_{4n+1} \to 0$, $|y_{4n+2}| \to \infty$ as $n \to \infty$.

(j) If $\left| \frac{x_{-1}y_{-k-1}}{y_{-2}x_{-k-2}} \frac{1}{bx_{-1}y_{-k-1}-1} \right| > 1$, then $|x_{4n+1}| \to \infty$, $y_{4n+2} \to 0$ as $n \to \infty$.

(k) If $\frac{x_{-1}y_{-k-1}}{y_{-2}x_{-k-2}} \frac{1}{bx_{-1}y_{-k-1}-1} = 1$, then x_{4n+1} and y_{4n+2} are $2k$-periodic.

(l) If $\frac{x_{-1}y_{-k-1}}{y_{-2}x_{-k-2}} \frac{1}{bx_{-1}y_{-k-1}-1} = -1$, then x_{4n+1} and y_{4n+2} are $4k$-periodic.

(m) If $\left| \frac{x_{-1}y_{-k-1}}{y_{-2}x_{-k-2}} \left(\beta y_{-2}x_{-k-2} - 1 \right) \right| < 1$, then $x_{4n+3} \to 0$, $|y_{4n}| \to \infty$ as $n \to \infty$.

(n) If $\left| \frac{x_{-1}y_{-k-1}}{y_{-2}x_{-k-2}} \left(\beta y_{-2}x_{-k-2} - 1 \right) \right| > 1$, then $|x_{4n+3}| \to \infty$, $y_{4n} \to 0$ as $n \to \infty$.

(o) If $\frac{x_{-1}y_{-k-1}}{y_{-2}x_{-k-2}} \left(\beta y_{-2}x_{-k-2} - 1 \right) = 1$, then x_{4n+3} and y_{4n} are $2k$-periodic.

(p) If $\frac{x_{-1}y_{-k-1}}{y_{-2}x_{-k-2}} \left(\beta y_{-2}x_{-k-2} - 1 \right) = -1$, then x_{4n+3} and y_{4n} are $4k$-periodic.

Proof From (2.3), in this case, we have

$$u_n = -u_{n-2} + b = u_{n-4}, \ v_n = -v_{n-2} + \beta = v_{n-4}, \ n \in \mathbb{N}_0, \ n \geq 2, \tag{3.72}$$

which means that the sequences u_n and v_n are four-periodic, and consequently the sequences $x_n y_{n-k}$ and $y_n x_{n-k}$ are four-periodic. Hence, we have

$$
\begin{aligned}
x_{4n}y_{4n-k} &= x_0 y_{-k}, \\
x_{4n+1}y_{4n-k+1} &= x_1 y_{-k+1}, \\
x_{4n+2}y_{4n-k+2} &= x_{-2} y_{-k-2}, \\
x_{4n+3}y_{4n-k+3} &= x_{-1} y_{-k-1},
\end{aligned}
\tag{3.73}
$$

and

$$
\begin{aligned}
y_{4n}x_{4n-k} &= y_0 x_{-k}, \\
y_{4n+1}x_{4n-k+1} &= y_1 x_{-k+1}, \\
y_{4n+2}x_{4n-k+2} &= y_{-2} x_{-k-2}, \\
y_{4n+3}x_{4n-k+3} &= y_{-1} x_{-k-1}.
\end{aligned}
\tag{3.74}
$$

If $k = 2t + 1$, for some $t \in \mathbb{N}_0$, then from (3.1), (3.73), and (3.74) we have

$$
\begin{aligned}
x_{4n}y_{4n-(2t+1)} &= x_0 y_{-(2t+1)}, \\
x_{4n+1}y_{4n-2t} &= x_1 y_{-2t}, \\
x_{4n+2}y_{4n-(2t-1)} &= x_{-2} y_{-(2t+3)}, \\
x_{4n+3}y_{4n-(2t-2)} &= x_{-1} y_{-(2t+2)},
\end{aligned}
\tag{3.75}
$$

and

$$
\begin{aligned}
y_{4n}x_{4n-(2t+1)} &= y_0 x_{-(2t+1)}, \\
y_{4n+1}x_{4n-2t} &= y_1 x_{-2t}, \\
y_{4n+2}x_{4n-(2t-1)} &= y_{-2} x_{-(2t+3)}, \\
y_{4n+3}x_{4n-(2t-2)} &= y_{-1} x_{-(2t+2)}.
\end{aligned}
\tag{3.76}
$$

Hence,

$$
\begin{aligned}
x_{4n}y_{4n-(2t+1)} &= x_0 y_{-(2t+1)} = \frac{x_{-2}y_{-(2t+3)}}{bx_{-2}y_{-(2t+3)} - 1}, \\
x_{4n+1}y_{4n-2t} &= x_1 y_{-2t} = \frac{x_{-1}y_{-(2t+2)}}{bx_{-1}y_{-(2t+2)} - 1},
\end{aligned}
\tag{3.77}
$$

and

$$
\begin{aligned}
y_{4n}x_{4n-(2t+1)} &= y_0 x_{-(2t+1)} = \frac{y_{-2}x_{-(2t+3)}}{\beta y_{-2}x_{-(2t+3)} - 1}, \\
y_{4n+1}x_{4n-2t} &= y_1 x_{-2t} = \frac{y_{-1}x_{-(2t+2)}}{\beta y_{-1}x_{-(2t+2)} - 1}.
\end{aligned}
\tag{3.78}
$$

From (3.75)–(3.78) we obtain

$$
x_{4n} = \frac{x_{-2}y_{-(2t+3)}}{bx_{-2}y_{-(2t+3)} - 1}\frac{1}{y_{4n-(2t+1)}} = \frac{x_{-2}y_{-(2t+3)}}{y_{-1}x_{-(2t+2)}}\frac{1}{bx_{-2}y_{-(2t+3)} - 1}x_{4n-(4t+2)},
\tag{3.79}
$$

if t is even,

$$
x_{4n+1} = \frac{x_{-1}y_{-(2t+2)}}{bx_{-1}y_{-(2t+2)} - 1}\frac{1}{y_{4n-2t}} = \frac{x_{-1}y_{-(2t+2)}}{y_{-2}x_{-(2t+3)}}\frac{1}{bx_{-1}y_{-(2t+2)} - 1}x_{4n+1-(4t+2)},
\tag{3.80}
$$

if t is odd,

$$
x_{4n+2} = \frac{x_{-2}y_{-(2t+3)}}{y_{4n-(2t-1)}} = \frac{x_{-2}y_{-(2t+3)}}{y_{-1}x_{-(2t+2)}}\left(\beta y_{-1}x_{-(2t+2)} - 1\right)x_{4n+2-(4t+2)},
\tag{3.81}
$$

if t is even,

$$
x_{4n+3} = \frac{x_{-1}y_{-(2t+2)}}{y_{4n-(2t-2)}} = \frac{x_{-1}y_{-(2t+2)}}{y_{-2}x_{-(2t+3)}}\left(\beta y_{-2}x_{-(2t+3)} - 1\right)x_{4n+3-(4t+2)},
\tag{3.82}
$$

if t is odd,

$$
y_{4n} = \frac{y_{-2}x_{-(2t+3)}}{\beta y_{-2}x_{-(2t+3)} - 1}\frac{1}{x_{4n-(2t+1)}} = \frac{y_{-2}x_{-(2t+3)}}{x_{-1}y_{-(2t+2)}}\frac{1}{\beta y_{-2}x_{-(2t+3)} - 1}y_{4n-(4t+2)},
\tag{3.83}
$$

if t is even,

$$
y_{4n+1} = \frac{y_{-1}x_{-(2t+2)}}{\beta y_{-1}x_{-(2t+2)} - 1}\frac{1}{x_{4n-2t}} = \frac{y_{-1}x_{-(2t+2)}}{x_{-2}y_{-(2t+3)}}\frac{1}{\beta y_{-1}x_{-(2t+2)} - 1}y_{4n+1-(4t+2)},
\tag{3.84}
$$

if t is odd,

$$
y_{4n+2} = \frac{y_{-2}x_{-(2t+3)}}{x_{4n-(2t-1)}} = \frac{y_{-2}x_{-(2t+3)}}{x_{-1}y_{-(2t+2)}}\left(bx_{-1}y_{-(2t+2)} - 1\right)y_{4n+2-(4t+2)},
\tag{3.85}
$$

if t is even,

$$
y_{4n+3} = \frac{y_{-1}x_{-(2t+2)}}{x_{4n-(2t-2)}} = \frac{y_{-1}x_{-(2t+2)}}{x_{-2}y_{-(2t+3)}}\left(bx_{-2}y_{-(2t+3)} - 1\right)y_{4n+3-(4t+2)}.
\tag{3.86}
$$

if t is odd. From relations (3.79)–(3.86), the results in this theorem can be easily seen. $\qquad\square$

Acknowledgement

References

[1] Ahmed AM, Elsayed EM. The expressions of solutions and the periodicity of some rational difference equations systems. Journal of Applied Mathematics and Informatics 2016; 34 (1-2): 35-48.

[2] Alzahrani EO, El-Dessoky MM, Elsayed EM, Kuang Y. Solutions and properties of some degenerate systems of difference equations. Journal of Computational Analysis and Applications 2015; 18: 321-333.

[3] Andruch A, Migda SM. Further properties of the rational recursive sequence $x_{n+1} = \frac{a x_{n-1}}{b + c x_n x_{n-1}}$. Opuscula Mathematica 2006; 26: 387-394.

[4] Cinar C. On the positive solutions of difference equation $x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}$. Applied Mathematics and Computation 2004; 150 (1): 21-24.

[5] Dekkar I, Touafek N, Yazlik Y. Global stability of a third-order nonlinear system of difference equations with period-two coefficients. Revista de la Real Academia 2017; 111: 325-347.

[6] Din Q. On a system of fourth-order rational difference equations. Acta Universitatis Apulensis Mathematics Informatics 2014; 39: 137-150.

[7] Elaydi S. An Introduction to Difference Equations. New York, NY, USA: Springer, 1996.

[8] El-Dessoky MM, Elsayed EM, Alghamdi M. Solutions and periodicity for some systems of fourth order rational difference equations. Journal Computational Analysis and Applications 2015; 18 (1): 179-194.

[9] El-Metwally H, Elsayed EM. Qualitative study of solutions of some difference equations. Abstract and Applied Analysis 2012; 2012: 248291.

[10] El-Metwally H, Elsayed EM. Solution and behavior of a third rational difference equation. Utilitas Mathematica 2012; 88: 27-42.

[11] El-Metwally H, Elsayed EM. Qualitative behavior of some rational difference equations. Journal Compututational Analysis and Applications 2016; 20 (2): 226-236.

[12] Elsayed EM. Solution for systems of difference equations of rational form of order two. Computational and Applied Mathematics 2014; 33: 751-765.

[13] Elsayed EM. Expression and behavior of the solutions of some rational recursive sequences. Mathematical Methods in the Applied Sciences 2016; 39: 5682-5694.

[14] Elsayed EM, Alshabi KN, Alzahrani F. Qualitative study of solution of some higher order difference equations. Journal of Computational Analysis And Applications 2019; 26 (7): 1179-1191.

[15] Elsayed EM, El-Metwally H. On the solutions of some nonlinear systems of difference equations. Advances in Difference Equations 2013; 161 (1): 1-14.

[16] Grove EA, Ladas G. Periodicities in Nonlinear Difference Equations Advances in Discrete Mathematics and Applications. Vol. 4. London, UK: Chapman & Hall/CRC, London, 2005.

[17] Haddad N, Touafek N, Rabago, JFT. Well-defined solutions of a system of difference equations. Journal of Applied Mathematics and Computing 2018; 56 (1-2): 439-458.

[18] Halim Y, Touafek N, Yazlık Y. Dynamic behavior of a second-order nonlinear rational difference equation. Turkish Journal of Mathematics 2015; 39 (6): 1004-1018.

[19] Ibrahim TF. On the third order rational difference equation $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}$. International Journal of Contemporary Mathematical Sciences 2009; 4: 1321-1334.

[20] Ibrahim TF, Touafek N. On a third order rational difference equation with variable coefficients. Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms 2013; 20: 251-264.

[21] Khaliq A, Elsayed EM. The dynamics and solution of some difference equations. Journal of Nonlinear Science and Applications 2016; 9: 1052-1063.

[22] Khan AQ, Din Q, Qureshi MN, Ibrahim TF. Global behavior of an anti-competitive system of fourth-order rational difference equations. Computational Ecology and Software 2014; 4 (1): 35-46.

[23] Kulenovic MRS. Discrete Dynamical Systems and Difference Equations with Mathematica. London, UK: Chapman & Hall/CRC Press, 2002.

[24] Kurbanli AS, Cinar C, Yalcinkaya I. On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1}+1}, y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1}+1}$. Mathematical and Computer Modelling 2011; 53: 1261-1267.

[25] Öcalan Ö. Oscillation of nonlinear difference equations with several coefficients. Communications in Mathematical Analysis 2008; 4 (1): 35–44.

[26] Okumus I, Soykan Y. Dynamical behavior of a system of three-dimensional nonlinear difference equations. Advances in Difference Equations 2018; 2018 (223): 1-15.

[27] Özban AY. On the positive solutions of the system of rational difference equations $x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}$. Journal of Mathematical Analysis and Applications 2006; 323: 26-32.

[28] Papaschinopoulos G, Schinas CJ. On the behavior of the solutions of a system of two nonlinear difference equations. Communications on Applied Nonlinear Analysis 1998; 5 (2): 47-59.

[29] Papaschinopoulos G, Stefanidou G. Asymptotic behavior of the solutions of a class of rational difference equations. International Journal of Difference Equations 2010; 5 (2): 233-249.

[30] Rabago JFT, Bacani JB. On a nonlinear difference equations. Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 2017; 24: 375-394.

[31] Raouf A. Global behavior of the rational Riccati difference equation of order two: the general case. Journal of Difference Equations and Applications 2012; 18: 947-961.

[32] Stević S. On a solvable rational system of difference equations. Applied Mathematics and Computation 2012; 219: 2896-2908.

[33] Stević S. On the system $x_{n+1} = \frac{y_n x_{n-k}}{y_{n-k+1}\left(a_n+b_n y_n x_{n-k}\right)}, \ y_{n+1} = \frac{x_n y_{n-k}}{x_{n-k+1}\left(c_n+d_n x_n y_{n-k}\right)}$. Applied Mathematics and Computation 2013; 219: 4526-4534.

[34] Stević S, Alghamdi MA, Alotaibi A, Elsayed EM. On a class of solvable higher-order difference equations. Filomat 2017; 31 (2): 461-477.

[35] Stević S, Diblík J, Iričanin B, Šmarda Z. On some solvable difference equations and systems of difference equations. Abstract and Applied Analysis 2012; 2012: 541761.

[36] Stević S, Diblík J, Iričanin B, Šmarda Z. Solvability of nonlinear difference equations of fourth order. Electronic Journal of Differential Equations 2014; 2014 (264): 1-14.

[37] Stević S, Iričanin B, Šmarda Z. On a close to symmetric system of difference equations of second order. Advances in Difference Equations 2015; 2015 (264): 1-17.

[38] Tollu DT, Yazlik Y, Taşkara N. On the solutions of two special types of Riccati difference equation via Fibonacci numbers. Advances in Difference Equations 2013; 2013: 174.

[39] Tollu DT, Yazlik Y, Taşkara N. On fourteen solvable systems of difference equations. Applied Mathematics and Computation 2014; 233: 310-319.

[40] Tollu DT, Yazlık Y, Taşkara N. On a solvable nonlinear difference equation of higher order. Turkish Journal of Mathematics 2018; 42: 1765-1778.

[41] Touafek N, Elsayed EM. On the periodicity of some systems of nonlinear difference equations. Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie 2012; 55 (103): 217-224.

[42] Yalcinkaya I. On the difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$. Discrete Dynamics in Nature and Society 2008; 2008: 805460.

[43] Yang X. On the system of rational difference equations $x_n = A + \frac{y_{n-1}}{x_{n-p}y_{n-q}}, y_n = A + \frac{x_{n-1}}{x_{n-r}y_{n-s}}$. Journal of Mathematical Analysis and Applications 2006; 307: 305-311.

[44] Yazlik Y. On the solutions and behavior of rational difference equations. Journal of Computational Analysis and Applications 2014; 17 (3): 584-594.

[45] Yazlik Y, Elsayed EM, Taşkara N. On the behavior of the solutions of difference equation systems. Journal of Computational Analysis and Applications 2014; 16 (5): 932-941.

[46] Yazlik Y, Tollu DT, Taşkara N. On the behavior of solutions for some systems of difference equations. Journal of Computational Analysis and Applications 2015; 18 (1): 166-178.

[47] Yazlik Y, Tollu DT, Taşkara N. On the solutions of a max-type difference equation system. Mathematical Methods in the Applied Sciences 2015; 38 (17): 4388-4410.

[48] Yazlik Y, Tollu DT, Taşkara N. On the solutions of a three-dimensional system of difference equations. Kuwait Journal of Sciences 2016; 43 (1): 95-111.

[49] Zayed EME, El-Moneam MA. On the rational recursive sequence $x_{n+1} = \frac{Ax_n + (\beta x_n + \gamma x_{n-k})}{Bx_n + Cx_{n-k}}$. Communications on Applied Nonlinear Analysis 2009; 16: 91-106.