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The $m-$Extension of Fibonacci and Lucas $p-$Difference Sequences

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Abstract. In this paper we define the $m-$extension of Fibonacci and Lucas $p-$difference sequences by using the $m-$extension of Fibonacci and Lucas $p-$numbers. We investigate some properties of our new sequences and introduce some relations between the $m-$extension of Fibonacci and Lucas $p-$difference sequences and the $m-$extension of Fibonacci and Lucas $p-$numbers. Moreover, we present the sums and generating function of the $m-$extension of Fibonacci and Lucas $p-$difference sequences. Finally, we study the $m-$extension of Fibonacci $p-$difference Newton polynomial interpolation.

1. Introduction

Fibonacci and Lucas numbers are the most popular and fascinating sequences in mathematics and related fields [1]. The classical Fibonacci and Lucas numbers are defined by $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$, for $n \in \mathbb{N}_0$, with the initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively. One of the important mathematical discovery of the modern Golden Section and Fibonacci and Lucas numbers theory is Fibonacci and Lucas $p-$numbers. Stakhov and Rozin presented the Fibonacci and Lucas $p-$sequence by the recurrence relations

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \tag{1}$$

and

$$L_p(n) = L_p(n-1) + L_p(n-p-1) \tag{2}$$

with the initial conditions, $F_p(0) = 0, F_p(1) = 1, \ldots, F_p(p) = 1$ and $L_p(0) = p+1, L_p(1) = 1, \ldots, L_p(p) = 1$, respectively, in [3]. Later on, Kocer et al. defined the $m-$extension of the Fibonacci and Lucas $p-$numbers,

$$F_{p,m}(n+p+1) = mF_{p,m}(n+p) + F_{p,m}(n) \tag{3}$$

and

$$L_{p,m}(n+p+1) = mL_{p,m}(n+p) + L_{p,m}(n), \tag{4}$$

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with the initial conditions $F_{p,m}(0) = 0, F_{p,m}(1) = 1, F_{p,m}(2) = m, F_{p,m}(3) = m^2, \dots, F_{p,m}(p+1) = m^p$ and $L_{p,m}(0) = p+1, L_{p,m}(1) = m, L_{p,m}(2) = m^2, L_{p,m}(3) = m^3, \dots, L_{p,m}(p+1) = m^{p+1}$, where p and n are non-negative integers and m is a positive real number [4]. Falcon and Plaza introduced the $k-$Fibonacci numbers that generalize both the classical Fibonacci and the Pell sequences [5]. Moreover, the $k-$Fibonacci numbers are the special case of the $m-$extension of Fibonacci $p-$numbers for $p = 1$ and $k = m$. After that, Falcon applied the concept of the finite difference to the $k-$Fibonacci numbers and introduced the $k-$Fibonacci difference sequence [2]. Also in the literature, it can be found more research works related with difference sequences such that [6] and [7].

The aim of this study is to extend the $k-$Fibonacci difference sequence into the $m-$extension of Fibonacci $p-$difference sequence. In addition, we define the $m-$extension of Lucas $p-$difference sequence and give some properties of these sequences.

2. The $m-$extension of Fibonacci and Lucas $p-$difference sequences

In this section we apply the finite difference to the $m-$extension of Fibonacci and Lucas $p-$numbers and call the $m-$extension of Fibonacci and Lucas $p-$difference sequences. If we apply the finite difference to the $m-$extension of Fibonacci and Lucas $p-$sequence $F_{p,m} = \{F_{p,m,n}\}$ and $L_{p,m} = \{L_{p,m,n}\}$, then we obtain $\Delta(F_{p,m,n}) = F_{p,m,n+1} - F_{p,m,n}$ and $\Delta(L_{p,m,n}) = L_{p,m,n+1} - L_{p,m,n}$. Moreover, $\Delta^2(F_{p,m,n}) = \Delta(\Delta(F_{p,m,n})) = F_{p,m,n+2} - 2F_{p,m,n+1} + F_{p,m,n}$ and $\Delta^2(L_{p,m,n}) = \Delta(\Delta(L_{p,m,n})) = L_{p,m,n+2} - 2L_{p,m,n+1} + L_{p,m,n}$. More general, we can write

$$\Delta^{(i)}(F_{p,m}) = F^{(i)}_{p,m} = \{\Delta^{(i)}(F_{p,m,n})\} = \{F^{(i)}_{p,m,n}\} = \{F^{(i-1)}_{p,m,n+1} - F^{(i-1)}_{p,m,n}\} \tag{5}$$

and

$$\Delta^{(i)}(L_{p,m}) = L^{(i)}_{p,m} = \{\Delta^{(i)}(L_{p,m,n})\} = \{L^{(i)}_{p,m,n}\} = \{L^{(i-1)}_{p,m,n+1} - L^{(i-1)}_{p,m,n}\}. \tag{6}$$

If $i = 0$, then $F^{(0)}_{p,m,n} = F_{p,m,n}$ and $L^{(0)}_{p,m,n} = L_{p,m,n}$. If $i = 1$, it is $\Delta^1 = \Delta$. Note that, to avoid cumbersome notations we denote both $F_{p,m,n}$ and $L_{p,m,n}$ by

$$\gamma_{p,m,n} = \begin{cases} F_{p,m,n} & , \text{if } \gamma_{p,m,0} = 0, \gamma_{p,m,1} = 1, \dots, \gamma_{p,m,n} = m^{n-1} \\ L_{p,m,n} & , \text{if } \gamma_{p,m,0} = p+1, \gamma_{p,m,1} = 1, \dots, \gamma_{p,m,n} = m^{n}. \end{cases} \tag{7}$$

We define the $m-$extension of Fibonacci and Lucas $p-$difference sequences by using the definition of difference relation as

$$\gamma^{(i)}_{p,m,n} = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \gamma_{p,m,n+i-j}. \tag{8}$$

Note that the $m-$extension of Fibonacci and Lucas $p-$difference sequences are

$$F^{(i)}_{p,m,n} = \sum_{j=0}^{i} (-1)^j \binom{i}{j} F_{p,m,n+i-j} \tag{9}$$

and

$$L^{(i)}_{p,m,n} = \sum_{j=0}^{i} (-1)^j \binom{i}{j} L_{p,m,n+i-j}. \tag{10}$$

Next, we will prove the $m-$extension of Fibonacci and Lucas $p-$difference sequences verify also the following relation.

Lemma 2.1. *For $n \geq p$, the $m-$extension of Fibonacci and Lucas $p-$difference sequences satisfy the recurrence relation that is*

$$\gamma^{(i)}_{p,m,n+1} = m\gamma^{(i)}_{p,m,n} + \gamma^{(i)}_{p,m,n-p}. \tag{11}$$

Proof. We use induction to prove the Lemma. For $i = 1$,

$$\begin{aligned} \gamma^{(1)}_{p,m,n+1} &= \gamma_{p,m,n+2} - \gamma_{p,m,n+1} \\ &= \left(m\gamma_{p,m,n+1} + \gamma_{p,m,n-p+1}\right) - \left(m\gamma_{p,m,n} + \gamma_{p,m,n-p}\right) \\ &= m\left(\gamma_{p,m,n+1} - \gamma_{p,m,n}\right) + \left(\gamma_{p,m,n-p+1} - \gamma_{p,m,n-p}\right) \\ &= m\gamma^{(1)}_{p,m,n} + \gamma^{(1)}_{p,m,n-p}. \end{aligned}$$

Let us suppose this formula is true for i: $\gamma^{(i)}_{p,m,n+1} = m\gamma^{(i)}_{p,m,n} + \gamma^{(i)}_{p,m,n-p}$. Then:

$$\begin{aligned} \gamma^{(i+1)}_{p,m,n+1} &= \gamma^{(i)}_{p,m,n+2} - \gamma^{(i)}_{p,m,n+1} \\ &= \left(m\gamma^{(i)}_{p,m,n+1} + \gamma^{(i)}_{p,m,n-p+1}\right) - \left(m\gamma^{(i)}_{p,m,n} + \gamma^{(i)}_{p,m,n-p}\right) \\ &= m\left(\gamma^{(i)}_{p,m,n+1} - \gamma^{(i)}_{p,m,n}\right) + \left(\gamma^{(i)}_{p,m,n-p+1} - \gamma^{(i)}_{p,m,n-p}\right) \\ &= m\gamma^{(i+1)}_{p,m,n} + \gamma^{(i+1)}_{p,m,n-p}. \end{aligned}$$

□

Theorem 2.2. *For $n \geq 2p$, the $m-$extension of Fibonacci $p-$difference sequence satisfy the relation*

$$F^{(i)}_{p,m,n} = F_{p,m,n-p+1}F^{(i)}_{p,m,p} + \sum_{j=0}^{p-1} F_{p,m,n-p-j}F^{(i)}_{p,m,j}. \tag{12}$$

Proof. We prove the theorem by induction on n. For $n = 2p$, we obtain

$$\begin{aligned} F^{(i)}_{p,m,2p} &= mF^{(i)}_{p,m,2p-1} + F^{(i)}_{p,m,p-1} \\ &= m\left[mF^{(i)}_{p,m,2p-2} + F^{(i)}_{p,m,p-2}\right] + F^{(i)}_{p,m,p-1} \\ &= m^2F^{(i)}_{p,m,2p-2} + F^{(i)}_{p,m,p-1} + mF^{(i)}_{p,m,p-2} \\ &= m^3F^{(i)}_{p,m,2p-3} + F^{(i)}_{p,m,p-1} + mF^{(i)}_{p,m,p-2} + m^2F^{(i)}_{p,m,p-3} \\ &\;\;\vdots \\ &= m^pF^{(i)}_{p,m,p} + F^{(i)}_{p,m,p-1} + mF^{(i)}_{p,m,p-2} + \cdots + m^{p-2}F^{(i)}_{p,m,1} + m^{p-1}F^{(i)}_{p,m,0}. \end{aligned}$$

By considering the initial conditions of the $m-$extension of the Fibonacci $p-$numbers, we have

$$F^{(i)}_{p,m,2p} = F_{p,m,p+1}F^{(i)}_{p,m,p} + F_{p,m,1}F^{(i)}_{p,m,p-1} + F_{p,m,2}F^{(i)}_{p,m,p-2} + \cdots + F_{p,m,p-2}F^{(i)}_{p,m,2} + F_{p,m,p-1}F^{(i)}_{p,m,1} + F_{p,m,p}F^{(i)}_{p,m,0}.$$

Assume that Eq. (12) is true for $n \geq 2p+1$. Then we have

$$\begin{aligned} F^{(i)}_{p,m,n+1} &= mF^{(i)}_{p,m,n} + F^{(i)}_{p,m,n-p} \\ &= m\left[F_{p,m,n-p+1}F^{(i)}_{p,m,p} + F_{p,m,n-2p+1}F^{(i)}_{p,m,p-1} + \cdots + F_{p,m,n-p}F^{(i)}_{p,m,0}\right] \\ &\quad + F_{p,m,n-2p+1}F^{(i)}_{p,m,p} + F_{p,m,n-3p+1}F^{(i)}_{p,m,p-1} + \cdots + F_{p,m,n-2p}F^{(i)}_{p,m,0} \\ &= (mF_{p,m,n-p+1} + F_{p,m,n-2p+1})F^{(i)}_{p,m,p} + (mF_{p,m,n-2p+1} + F_{p,m,n-3p+1})F^{(i)}_{p,m,p-1} + \cdots \\ &\quad + (mF_{p,m,n-p} + F_{p,m,n-2p})F^{(i)}_{p,m,0} \\ &= F_{p,m,n-p+2}F^{(i)}_{p,m,p} + \sum_{j=0}^{p-1} F_{p,m,n+1-p-j}F^{(i)}_{p,m,j}, \end{aligned}$$

which completes the proof. □

Theorem 2.3. *For $i \geq 1$, $n \geq p+1$ and $m > 1$, the $m-$extension of Fibonacci and Lucas $p-$difference sequences satisfy the relation,*

$$\gamma^{(i)}_{p,m,n} = (m-1)\gamma^{(i-1)}_{p,m,n} + \gamma^{(i-1)}_{p,m,n-p} \tag{13}$$

with the initial condition $\gamma^{(i)}_{p,m,0} = \gamma^{(i-1)}_{p,m,1} - \gamma^{(i-1)}_{p,m,1-p}$.

Proof. From the definition of the $m-$extension of Fibonacci and Lucas $p-$difference sequences, we obtain

$$\gamma^{(i-1)}_{p,m,n-p} = \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\gamma_{p,m,n-p+i-j-1}$$

and

$$\gamma^{(i-1)}_{p,m,n} = \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\gamma_{p,m,n+i-j-1}.$$

Using these identities, we get

$$\begin{aligned} (m-1)\gamma^{(i-1)}_{p,m,n} + \gamma^{(i-1)}_{p,m,n-p} &= (m-1)\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\gamma_{p,m,n+i-j-1} + \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\gamma_{p,m,n-p+i-j-1} \\ &= \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}(m\gamma_{p,m,n+i-j-1} - \gamma_{p,m,n+i-j-1} + \gamma_{p,m,n-p+i-j-1}) \\ &= \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}(m\gamma_{p,m,n+i-j-1} + \gamma_{p,m,n-p+i-j-1}) - \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}(\gamma_{p,m,n+i-j-1}) \\ &= \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\gamma_{p,m,n+i-j} - \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\gamma_{p,m,n+i-j-1} \end{aligned}$$

$$
\begin{aligned}
&= \gamma_{p,m,n+i} + \sum_{j=1}^{i-1}(-1)^j\binom{i-1}{j}\gamma_{p,m,n+i-j} - \sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\gamma_{p,m,n+i-j-1} \\
&= \gamma_{p,m,n+i} + \sum_{j=1}^{i-1}(-1)^j\left[\binom{i-1}{j}+\binom{i-1}{j-1}\right]\gamma_{p,m,n+i-j} - (-1)^{i-1}\gamma_{p,m,n} \\
&= \gamma_{p,m,n+i} + \sum_{j=1}^{i-1}(-1)^j\binom{i}{j}\gamma_{p,m,n+i-j} + (-1)^i\gamma_{p,m,n} \\
&= \sum_{j=0}^{i}(-1)^j\binom{i}{j}\gamma_{p,m,n+i-j} \\
&= \gamma^{(i)}_{p,m,n}.
\end{aligned}
$$

□

Theorem 2.4. *Sum of the $m-$extension of Fibonacci and Lucas $p-$difference sequences is*

$$\sum_{j=0}^{n}\gamma^{(i)}_{p,m,j} = \gamma^{(i-1)}_{p,m,n+1} - \gamma^{(i-1)}_{p,m,0}. \tag{14}$$

Proof. By considering the $m-$extension of Fibonacci and Lucas numbers with negative subscript given in [4] and using Eq. (11), we have

$$
\begin{aligned}
\sum_{j=0}^{n}\gamma^{(i)}_{p,m,j} &= \gamma^{(i)}_{p,m,0} + \sum_{j=1}^{n}\gamma^{(i)}_{p,m,j} \\
&= \gamma^{(i)}_{p,m,0} + \frac{1}{m}\sum_{j=1}^{n}(\gamma^{(i)}_{p,m,j+1} - \gamma^{(i)}_{p,m,j-p}) \\
&= \gamma^{(i)}_{p,m,0} + \frac{1}{m}\left(\left(\gamma^{(i)}_{p,m,2} - \gamma^{(i)}_{p,m,1-p}\right) + \left(\gamma^{(i)}_{p,m,3} - \gamma^{(i)}_{p,m,2-p}\right) + \cdots + \left(\gamma^{(i)}_{p,m,n+1} - \gamma^{(i)}_{p,m,n-p}\right)\right) \\
&= \gamma^{(i)}_{p,m,0} + \frac{1}{m}\left(\left(\gamma^{(i-1)}_{p,m,3} - \gamma^{(i-1)}_{p,m,2} - \gamma^{(i-1)}_{p,m,2-p} + \gamma^{(i-1)}_{p,m,1-p}\right) + \cdots + \left(\gamma^{(i-1)}_{p,m,n+2} - \gamma^{(i-1)}_{p,m,n+1} - \gamma^{(i-1)}_{p,m,n-p+1} + \gamma^{(i-1)}_{p,m,n-p}\right)\right) \\
&= \gamma^{(i)}_{p,m,0} + \frac{1}{m}\left(\gamma^{(i-1)}_{p,m,n+2} - \gamma^{(i-1)}_{p,m,n+1-p} - \gamma^{(i-1)}_{p,m,2} + \gamma^{(i-1)}_{p,m,1-p}\right) \\
&= \gamma^{(i-1)}_{p,m,1} - \gamma^{(i-1)}_{p,m,0} + \gamma^{(i-1)}_{p,m,n+1} - \gamma^{(i-1)}_{p,m,1} \\
&= \gamma^{(i-1)}_{p,m,n+1} - \gamma^{(i-1)}_{p,m,0}.
\end{aligned}
$$

□

Theorem 2.5. *The generating function of the $m-$extension of Fibonacci and Lucas $p-$sequences is*

$$\gamma^{(i)}(x) = \frac{\gamma^{(i)}_{p,m,0} + \sum_{j=1}^{p}\left(\gamma^{(i)}_{p,m,j} - m\gamma^{(i)}_{p,m,j-1}\right)x^j}{1 - mx - x^{p+1}}. \tag{15}$$

Proof. Let $\gamma^{(i)}(x)$ be the generating function of the $m-$extension of Fibonacci and Lucas $p-$sequence $\gamma_{p,m,n}$. Then, we have

$$\begin{aligned}\gamma^{(i)}(x) &= \gamma^{(i)}_{p,m,0} + \gamma^{(i)}_{p,m,1}x + \gamma^{(i)}_{p,m,2}x^2 + \cdots + \gamma^{(i)}_{p,m,p}x^p + \gamma^{(i)}_{p,m,p+1}x^{p+1} + \cdots \\ m\gamma^{(i)}(x)x &= m\gamma^{(i)}_{p,m,0}x + m\gamma^{(i)}_{p,m,1}x^2 + m\gamma^{(i)}_{p,m,2}x^3 + \cdots + m\gamma^{(i)}_{p,m,p}x^{p+1} + m\gamma^{(i)}_{p,m,p+1}x^{p+2} + \cdots \\ \gamma^{(i)}(x)x^{p+1} &= \gamma^{(i)}_{p,m,0}x^{p+1} + \gamma^{(i)}_{p,m,1}x^{p+2} + \gamma^{(i)}_{p,m,2}x^{p+3} + \cdots + \gamma^{(i)}_{p,m,p}x^{2p+1} + \gamma^{(i)}_{p,m,p+1}x^{2p+2} + \cdots.\end{aligned}$$

By using these identities, we obtain

$$\begin{aligned}\left(1 - mx - x^{p+1}\right)\gamma^{(i)}(x) &= \gamma^{(i)}_{p,m,0} + \left(\gamma^{(i)}_{p,m,1} - m\gamma^{(i)}_{p,m,0}\right)x + \cdots + \left(\gamma^{(i)}_{p,m,p} - m\gamma^{(i)}_{p,m,p-1}\right)x^p \\ &= \gamma^{(i)}_{p,m,0} + \sum_{j=1}^{p}\left(\gamma^{(i)}_{p,m,j} - m\gamma^{(i)}_{p,m,j-1}\right)x^j. \qquad (16)\end{aligned}$$

Therefore, the generating function of the $m-$extension of Fibonacci and Lucas $p-$sequences is

$$\gamma^{(i)}(x) = \frac{\gamma^{(i)}_{p,m,0} + \sum_{j=1}^{p}\left(\gamma^{(i)}_{p,m,j} - m\gamma^{(i)}_{p,m,j-1}\right)x^j}{1 - mx - x^{p+1}}.$$

□

The next theorem gives the relation between $m-$ extension of Fibonacci and Lucas $p-$difference sequences.

Theorem 2.6. *For $n \geq p$ and $m > 1$, the $m-$extension of Fibonacci and Lucas $p-$difference numbers satisfy the relation*

$$L^{(i)}_{p,m,n} = F^{(i)}_{p,m,n+1} + pF^{(i)}_{p,m,n-p}. \qquad (17)$$

Proof. By using Eq. (2.5) in [8] and Eq. (8), we have

$$\begin{aligned}F^{(i)}_{p,m,n+1} + pF^{(i)}_{p,m,n-p} &= \sum_{j=0}^{i}(-1)^j\binom{i}{j}F_{p,m,n+1+i-j} + p\sum_{j=0}^{i}(-1)^j\binom{i}{j}F_{p,m,n-p+i-j} \\ &= \sum_{j=0}^{i}(-1)^j\binom{i}{j}\left[F_{p,m,n+1+i-j} + pF_{p,m,-p+i-j}\right] \\ &= \sum_{j=0}^{i}(-1)^j\binom{i}{j}L_{p,m,n+i-j} \\ &= L^{(i)}_{p,m,n}.\end{aligned}$$

□

Lemma 2.7. *For $r \leq i$ and $m > 1$, the following identity holds:*

$$\gamma^{(i)}_{p,m,n} = \sum_{j=0}^{r}\binom{r}{j}(m-1)^{r-j}\gamma^{(i-r)}_{n-jp}. \qquad (18)$$

3. $m-$extension of Fibonacci $p-$difference Newton interpolation

Let us consider the $(n+1)$ points $(x_j, F_{p,m,j})$, $j = 0, 1, 2, \ldots, n$ with $x_j < x_{j+1}$ and suppose we wish to find a polynomial $P_n(m, x)$ that takes the value $F_{p,m,j}$ for $x = x_j$. Now we define $h_j = x_{j+1} - x_j$. Therefore, the $m-$extension of Fibonacci $p-$Newton interpolation is

$$P_n(m,x) = F_{p,m,0} + F^{(1)}_{p,m,0}\frac{x-x_0}{h_0} + \frac{F^{(2)}_{p,m,0}}{2!}\frac{x-x_0}{h_0}\frac{x-x_1}{h_1} + \frac{F^{(3)}_{p,m,0}}{3!}\frac{x-x_0}{h_0}\frac{x-x_1}{h_1}\frac{x-x_2}{h_2} + \cdots, \tag{19}$$

or in reduced form,

$$P_n(m,x) = F_{p,m,0} + \sum_{i=1}^{n}\frac{F^{(i)}_{p,m,0}}{i!}\prod_{j=0}^{i-1}\frac{x-x_j}{h_j}. \tag{20}$$

The formula can be simplified if $x_j = j$ and takes the more practical form

$$P_n(m,x) = \sum_{i=1}^{n}\frac{F^{(i)}_{p,m,0}}{i!}\prod_{j=0}^{i-1}(x-j). \tag{21}$$

If $x_{j+1} - x_j = h$, $\forall j$, the error is given by

$$\varepsilon = \frac{F^{(n+1)}_{p,m,0}}{(n+1)!}\frac{1}{h^{n+1}}\prod_{j=0}^{n}(x-x_j). \tag{22}$$

Thus the maximum error will arise at some point in the interval between two consecutive nodes.

4. Examples

Let us suppose we have the four points $(x_j, F_{p,m,j})$, $j = 0, 1, 2, 3$. For $p = 1$ and $m = 3$, we can find the same results in [2]. We find the interpolation polynomials for higher p and different m values. For example, for $p = 10$, the $m-$extension of Fibonacci $p-$Newton interpolation is

$$\begin{aligned}
P_4(m,x) &= F_{p,m,0} + \frac{F^{(1)}_{p,m,0}}{1!}(x-x_0) \\
&\quad + \frac{F^{(2)}_{p,m,0}}{2!}(x-x_0)(x-x_1) + \frac{F^{(3)}_{p,m,0}}{3!}(x-x_0)(x-x_1)(x-x_2) \\
&\quad + \frac{F^{(4)}_{p,m,0}}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3) \\
&= x + \frac{1}{2}(2(m-1)-m)(x-1)x \\
&\quad + \frac{1}{6}\left(m^2 + 3(m-1)^2 - 3m(m-1)\right)(x-2)(x-1)x \\
&\quad + \frac{1}{24}\left(-m^3 + 4m^2(m-1)\right. \\
&\quad \left. + 4(m-1)^3 - 6m(m-1)^2\right)(x-3)(x-2)(x-1)x.
\end{aligned}$$

The following table represents five interpolation polynomials and error bounds for different values of m.

m	$P_4(m,x)$	ε
3	$\frac{1}{24}\left(5x^4-18x^3+31x^2+6x\right)$	$\frac{11}{120}\max\lvert(x-4)(x-3)(x-2)(x-1)x\rvert = 0.0325937$
4	$\frac{1}{6}\left(5x^4-23x^3+40x^2-16x\right)$	$\frac{61}{120}\max\lvert(x-4)(x-3)(x-2)(x-1)x\rvert = 0.180747$
5	$\frac{1}{24}\left(51x^4-254x^3+441x^2-214x\right)$	$\frac{41}{24}\max\lvert(x-4)(x-3)(x-2)(x-1)x\rvert = 0.607427$
6	$\frac{1}{6}\left(26x^4-135x^3+235x^2-120x\right)$	$\frac{521}{120}\max\lvert(x-4)(x-3)(x-2)(x-1)x\rvert = 1.54375$
7	$\frac{1}{24}\left(185x^4-986x^3+1723x^2-898x\right)$	$\frac{1111}{120}\max\lvert(x-4)(x-3)(x-2)(x-1)x\rvert = 3.29196$

If we apply directly the classical Newton interpolation for $p = 10$ and $m = 5$, we obtain the following interpolation table.

x_j	$F_{p,m,j}$	$\Delta F_{p,m,j}$	$\Delta^2 F_{p,m,j}$	$\Delta^3 F_{p,m,j}$	$\Delta^4 F_{p,m,j}$
0	0				
		1			
1	1		3		
		4		13	
2	5		16		51
		20		64	
3	25		80		
		100			
4	125				

As seen from the table, for any p or m, the numbers of the first diagonal line are the numerators of the coefficients of the Newton polynomial that

$$P_4(5,x) = 0 + \frac{1}{1!}x + \frac{3}{2!}x(x-1) + \frac{13}{3!}x(x-1)(x-2) + \frac{51}{4!}x(x-1)(x-2)(x-3).$$

5. Conclusion

In this study, for the first time in the literature, we define the $m-$extension of Fibonacci and Lucas $p-$difference sequence by using the $m-$extension of Fibonacci and Lucas $p-$numbers. If we take $p = 1$ and $m = k$, this sequence can be reduced to $k-$Fibonacci difference sequence [2]. If we take $p = 1$ and $m = 2$, this sequence can be reduced to Pell difference sequence [6]. We give generating functions, sums and some properties for this sequence. Moreover, we apply newton interpolation and present the results with interpolation table. As a result, this study contributes to the literature by providing essential information for the generalization of difference sequences.

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