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A SOLVABLE SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. In this paper, we show that the system of difference equations

$$x_n = \frac{a y_{n-1}^p + b\left(x_{n-2} y_{n-1}\right)^{p-1}}{c y_{n-1} + d x_{n-2}^{p-1}},\ y_n = \frac{\alpha x_{n-1}^p + \beta\left(y_{n-2} x_{n-1}\right)^{p-1}}{\gamma x_{n-1} + \delta y_{n-2}^{p-1}},$$

$n \in \mathbb{N}_0$ where the parameters $a, b, c, d, \alpha, \beta, \gamma, \delta, p$ and the initial values $x_{-2},\ x_{-1},\ y_{-2},\ y_{-1}$ are real numbers, can be solved. Also, by using obtained formulas, we study the asymptotic behaviour of well-defined solutions of aforementioned system and describe the forbidden set of the initial values. Our obtained results significantly extend and develop some recent results in the literature.

1. Introduction and preliminaries

Studying solvability of non-linear difference equations and systems is a topic of a great interest (see, e.g. [1, 2, 4–6, 8, 9, 11, 12, 14–25, 27, 28] and as well as the references therein). This is probably due to the necessity of applying different methods for each type of non-linear equation. These methods are generally based on that a non-linear equation reduces to a linear equation, by using some suitable changes of variables. In the last decade, many researchers have worked on non-linear difference equations that can be solved. A well-known non-linear difference equation which can be solved is the equation

$$x_n = \frac{a x_{n-1} + b}{c x_{n-1} + d},\ n \in \mathbb{N}_0, \tag{1}$$

where initial value x_{-1} is a real number, which is called Riccati difference equation. In the literature, there are so many studies on Eq. (1) (see, for example [1, 9, 12, 14, 21, 23, 24]). In [15], Eq. (1) was generalized by McGrath and Teixeira to the following equation

$$x_n = \frac{a x_{n-1}^2 + b x_{n-2} x_{n-1}}{c x_{n-1} + d x_{n-2}},\ n \in \mathbb{N}_0, \tag{2}$$

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where the parameters a, b, c, d and the initial values x_{-2}, x_{-1} are real numbers. The authors solved Eq. (2) and investigated the existence and behavior of the solutions of Eq. (2) by using some known results. Also, [14], Eq. (1) was extended to the following two-dimensional system of difference equation

$$x_n = \frac{ay_{n-1}+b}{cy_{n-1}+d},\ y_n = \frac{ax_{n-1}+b}{cx_{n-1}+d},\ n \in \mathbb{N}_0, \tag{3}$$

where a, b, c, d are real numbers with $c \neq 0$ and $ad - bc \neq 0$. The solution formulas of the system (3) were proved by induction.

A natural problem is to extend a two dimensional relative of Eq. (2) solvable in closed form. In this paper, we will consider such a system. That is, we show that the following system of difference equations

$$x_n = \frac{ay^p_{n-1}+b(x_{n-2}y_{n-1})^{p-1}}{cy_{n-1}+dx^{p-1}_{n-2}},\ y_n = \frac{\alpha x^p_{n-1}+\beta(y_{n-2}x_{n-1})^{p-1}}{\gamma x_{n-1}+\delta y^{p-1}_{n-2}},\ n \in \mathbb{N}_0, \tag{4}$$

where the parameters $a, b, c, d, \alpha, \beta, \gamma$, δ, p and the initial values x_{-2}, x_{-1}, y_{-2}, y_{-1} are real numbers, can be solved. Also, by using the obtained formulas we study the asymptotic behaviour of well-defined solutions of system (4). Note that by using some transformation, the system (4) can be reduced to the system (3). The system (4) can be extended to the system

$$x_n = \frac{ay^p_{n-k}+b\big(x_{n-(k+l)}y_{n-k}\big)^{p-1}}{cy_{n-k}+dx^{p-1}_{n-(k+l)}},\ y_n = \frac{\alpha x^p_{n-l}+\beta\big(y_{n-(k+l)}x_{n-l}\big)^{p-1}}{\gamma x_{n-l}+\delta y^{p-1}_{n-(k+l)}},\ n \in \mathbb{N}_0. \tag{5}$$

However, to simplify the calculations, we restricted our work to the system (4).

It is not hard to see, that if in (5), we take, $p = 2$, $y_{-i} = x_{-i}$, $i = 0, \dots, k+l$ and a particular choice of the parameters a, b, c, d, α, β, γ, δ, then for $l = k$, we get a special case of the equation

$$x_n = ax_{n-k} + \frac{bx_{n-k}x_{n-(k+l)}}{cx_{n-l} + dx_{n-(k+l)}}.$$

The solutions of this last equation have been studied in [25]. Besides, there are also studies about dynamics of non-linear difference equations and systems (see [3, 10, 13, 26]). In the analysis of solutions of a difference equation or a system, the matter of existence of solutions is of prime importance as such in differential equations. Before giving our main results, we recall the following definition which states the set of initial values which yields undefinable solutions. In our investigation, we are inspired by the ideas and the technics of calculations presented in some of the references given in the end of this work, for example [7, 19, 20, 25].

Definition. Consider the following system of difference equations

$$x_n = f_1(x_{n-1}, x_{n-2}, y_{n-1}, y_{n-2}),\ y_n = f_2(x_{n-1}, x_{n-2}, y_{n-1}, y_{n-2}),\ n \in \mathbb{N}_0, \tag{6}$$

where the initial values $x_{-2}, x_{-1}, y_{-2}, y_{-1}$ are real numbers and D_1 and D_2 are domains of the functions f_1 and f_2, respectively. The forbidden set of system

(6) is given by

$$\mathcal{F} = \Big\{ (x_{-2}, x_{-1}, y_{-2}, y_{-1}) \in \mathbb{R}^4 : (x_i, y_i) \in D_1 \times D_2 \text{ for } i = 0, 1, \ldots, n-1, \\ \text{and } (x_n, y_n) \notin D_1 \times D_2 \Big\}.$$

2. Main results

In this section we prove our main results in which we give closed formulas for the well-defined solutions of the system (4). To start, we have the following observation.

Lemma 2.1. *Let* $\{(x_n, y_n)\}_{n \geq -2}$ *be a well-defined solution of the system* (4). *Then it satisfies*

$$x_n y_n \neq 0, \quad n \geq -1.$$

Proof. If we suppose that there exists $n_0 \geq -1$ such that $x_{n_0} = 0$, then we have

$$y_{n_0+1} = \frac{\alpha x_{n_0}^p + \beta (y_{n_0-1} x_{n_0})^{p-1}}{\gamma x_{n_0} + \delta y_{n_0-1}^{p-1}} = 0.$$

Therefore, the term x_{n_0+2} is undefinable. Similarly, if we suppose that there exists $n_0 \geq -1$, such that $y_{n_0} = 0$, then we have

$$x_{n_0+1} = \frac{a y_{n_0}^p + b (x_{n_0-1} y_{n_0})^{p-1}}{c y_{n_0} + d x_{n_0-1}^{p-1}} = 0.$$

Therefore, the term y_{n_0+2} is undefinable. □

As for the case when $x_{-2} = 0$ or $y_{-2} = 0$. First, consider the system (4) with $ac\alpha\gamma x_{-1} y_{-1} \neq 0$ and $x_{-2} y_{-2} = 0$. Then, we have one of the cases

$$x_0 = \frac{a}{c} y_{-1}^{p-1}, \ y_0 = \frac{\alpha x_{-1}^p + \beta y_{-2} x_{-1}^{p-1}}{\gamma x_{-1} + \delta y_{-2}^{p-1}},$$

$$x_0 = \frac{a}{c} y_{-1}^{p-1}, \ y_0 = \frac{\alpha}{\gamma} x_{-1}^{p-1}$$

or

$$x_0 = \frac{\alpha y_{-1}^p + \beta x_{-2} y_{-1}^{p-1}}{\gamma y_{-1} + \delta x_{-2}^{p-1}}, \ y_0 = \frac{\alpha}{\gamma} x_{-1}^{p-1}.$$

For the next terms, the condition $x_n y_n \neq 0$ is satisfied. Therefore, without loss of generality, we can suppose $x_{-2} y_{-2} \neq 0$.

2.1. Solvability of the system (4)

Consider the system (4) such that $x_{-2}y_{-2} \neq 0$. We rearrange the system (4) as follows:

$$\frac{x_n}{y_{n-1}^{p-1}} = \frac{a\frac{y_{n-1}}{x_{n-2}^{p-1}} + b}{c\frac{y_{n-1}}{x_{n-2}^{p-1}} + d}, \quad \frac{y_n}{x_{n-1}^{p-1}} = \frac{\alpha\frac{x_{n-1}}{y_{n-2}^{p-1}} + \beta}{\gamma\frac{x_{n-1}}{y_{n-2}^{p-1}} + \delta}, \ n \in \mathbb{N}_0. \tag{7}$$

Putting

$$u_n = \frac{x_n}{y_{n-1}^{p-1}}, \ v_n = \frac{y_n}{x_{n-1}^{p-1}}, \ n \geq -1, \tag{8}$$

we get

$$u_n = \frac{av_{n-1} + b}{cv_{n-1} + d}, \ v_n = \frac{\alpha u_{n-1} + \beta}{\gamma u_{n-1} + \delta}, \ n \in \mathbb{N}_0. \tag{9}$$

So

$$u_n = \frac{(a\alpha + b\gamma)\, u_{n-2} + a\beta + b\delta}{(c\alpha + d\gamma)\, u_{n-2} + c\beta + d\delta}, \ v_n = \frac{(a\alpha + c\beta)\, v_{n-2} + b\alpha + d\beta}{(a\gamma + c\delta)\, v_{n-2} + b\gamma + d\delta}, \ n \in \mathbb{N}. \tag{10}$$

If we apply the decomposition of indices $n \to 2m + i, i \in \{-1, 0\}$, to (10), then it becomes

$$u_{2m+i} = \frac{(a\alpha + b\gamma)\, u_{2(m-1)+i} + a\beta + b\delta}{(c\alpha + d\gamma)\, u_{2(m-1)+i} + c\beta + d\delta}, \ v_{2m+i} = \frac{(a\alpha + c\beta)\, v_{2(m-1)+i} + b\alpha + d\beta}{(a\gamma + c\delta)\, v_{2(m-1)+i} + b\gamma + d\delta}, \tag{11}$$

$m \in \mathbb{N}$, which are first-order 2−equations. Let $u_{2m+i} = u_m^{(i)}$, $v_{2m+i} = v_m^{(i)}$ for $m \in \mathbb{N}_0$ and $i \in \{-1, 0\}$. Then, equations in (11) can be written as the following

$$u_m^{(i)} = \frac{(a\alpha + b\gamma)\, u_{m-1}^{(i)} + a\beta + b\delta}{(c\alpha + d\gamma)\, u_{m-1}^{(i)} + c\beta + d\delta}, \ m \in \mathbb{N}, \tag{12}$$

$$v_m^{(i)} = \frac{(a\alpha + c\beta)\, v_{m-1}^{(i)} + b\alpha + d\beta}{(a\gamma + c\delta)\, v_{m-1}^{(i)} + b\gamma + d\delta}, \ m \in \mathbb{N}, \tag{13}$$

which is essentially in the form of Riccati difference equation. Suppose that

$$c\alpha + d\gamma, a\gamma + c\delta \neq 0$$

and

$$(a\alpha + b\gamma)(c\beta + d\delta) - (a\beta + b\delta)(c\alpha + d\gamma) \neq 0,$$
$$(a\alpha + c\beta)(b\gamma + d\delta) - (a\gamma + c\delta)(b\alpha + d\beta) \neq 0.$$

If we use the change of variables

$$u_m^{(i)} = \frac{a\alpha + b\gamma + c\beta + d\delta}{c\alpha + d\gamma} r_m - \frac{c\beta + d\delta}{c\alpha + d\gamma}, \ m \in \mathbb{N}_0, \tag{14}$$

in Eq. (12), and

$$v_m^{(i)} = \frac{a\alpha + b\gamma + c\beta + d\delta}{a\gamma + c\delta} s_m - \frac{b\gamma + d\delta}{a\gamma + c\delta}, \ m \in \mathbb{N}_0, \tag{15}$$

in Eq. (13), then equations in (12) and (13) are transformed into the following equations

$$r_m = \frac{-R + r_{m-1}}{r_{m-1}}, \ s_m = \frac{-R + s_{m-1}}{s_{m-1}}, \ m \in \mathbb{N}, \tag{16}$$

where $R = \frac{(bc-ad)(\beta\gamma-\alpha\delta)}{(a\alpha+b\gamma+c\beta+d\delta)^2}$. The equations in (16) can be transformed into the following equations

$$z_{m+1} = z_m - Rz_{m-1}, \ m \in \mathbb{N}, \tag{17}$$

and

$$\widetilde{z}_{m+1} = \widetilde{z}_m - R\widetilde{z}_{m-1} \ m \in \mathbb{N}, \tag{18}$$

by means of the change of variables $r_m = \frac{z_{m+1}}{z_m}$ with the initial values $z_0 = 1$ and $z_1 = r_0$ and $s_m = \frac{\widetilde{z}_{m+1}}{\widetilde{z}_m}$ with the initial values $\widetilde{z}_0 = 1$ and $\widetilde{z}_1 = s_0$, respectively. If λ_1 and λ_2 are the complex roots of the characteristic equation of (17) and (18), which has the form $\lambda^2 - \lambda + R = 0$, the general solutions of equations in (17) and (18) are

$$z_m = \left(\frac{r_0 - \lambda_2}{\lambda_1 - \lambda_2}\right)\lambda_1^m + \left(\frac{\lambda_1 - r_0}{\lambda_1 - \lambda_2}\right)\lambda_2^m, \ m \in \mathbb{N}_0, \tag{19}$$

$$\widetilde{z}_m = \left(\frac{s_0 - \lambda_2}{\lambda_1 - \lambda_2}\right)\lambda_1^m + \left(\frac{\lambda_1 - s_0}{\lambda_1 - \lambda_2}\right)\lambda_2^m, \ m \in \mathbb{N}_0, \tag{20}$$

when $1 - 4R \neq 0$, and

$$z_m = (1 + (2r_0 - 1)\,m)\left(\frac{1}{2}\right)^m, \ m \in \mathbb{N}_0, \tag{21}$$

$$\widetilde{z}_m = (1 + (2s_0 - 1)\,m)\left(\frac{1}{2}\right)^m, \ m \in \mathbb{N}_0, \tag{22}$$

when $1 - 4R = 0$. By substituting (19) and (21) into $r_m = \frac{z_{m+1}}{z_m}$, (20) and (22) into $s_m = \frac{\widetilde{z}_{m+1}}{\widetilde{z}_m}$ respectively, we get

$$r_m = \frac{(r_0 - \lambda_2)\,\lambda_1^{m+1} + (\lambda_1 - r_0)\,\lambda_2^{m+1}}{(r_0 - \lambda_2)\,\lambda_1^m + (\lambda_1 - r_0)\,\lambda_2^m}, \ m \in \mathbb{N}_0, \tag{23}$$

$$s_m = \frac{(s_0 - \lambda_2)\,\lambda_1^{m+1} + (\lambda_1 - s_0)\,\lambda_2^{m+1}}{(s_0 - \lambda_2)\,\lambda_1^m + (\lambda_1 - s_0)\,\lambda_2^m}, \ m \in \mathbb{N}_0, \tag{24}$$

when $R \neq \frac{1}{4}$, and

$$r_m = \frac{1 + (2r_0 - 1)\,(m + 1)}{2 + (4r_0 - 2)\,m}, \ m \in \mathbb{N}_0, \tag{25}$$

$$s_m = \frac{1 + (2s_0 - 1)(m+1)}{2 + (4s_0 - 2)m},\ m \in \mathbb{N}_0, \tag{26}$$

when $R = \frac{1}{4}$. Consequently,

$$u_m^{(i)} = \frac{A}{B_1} \frac{\left(B_1 u_0^{(i)} + C_1 - \lambda_2 A\right)\lambda_1^{m+1} + \left(A\lambda_1 - B_1 u_0^{(i)} - C_1\right)\lambda_2^{m+1}}{\left(B_1 u_0^{(i)} + C_1 - \lambda_2 A\right)\lambda_1^{m} + \left(A\lambda_1 - B_1 u_0^{(i)} - C_1\right)\lambda_2^{m}} - \frac{C_1}{B_1}, \tag{27}$$

$$v_m^{(i)} = \frac{A}{B_2} \frac{\left(B_2 v_0^{(i)} + C_2 - \lambda_2 A\right)\lambda_1^{m+1} + \left(A\lambda_1 - B_2 v_0^{(i)} - C_2\right)\lambda_2^{m+1}}{\left(B_2 v_0^{(i)} + C_2 - A\lambda_2\right)\lambda_1^{m} + \left(A\lambda_1 - B_2 v_0^{(i)} - C_2\right)\lambda_2^{m}} - \frac{C_2}{B_2}, \tag{28}$$

when $R \neq \frac{1}{4}$, and

$$u_m^{(i)} = \frac{A}{B_1}\left(\frac{A + \left(2B_1 u_0^{(i)} + 2C_1 - A\right)(m+1)}{2A + \left(4B_1 u_0^{(i)} + 4C_1 - 2A\right)m}\right) - \frac{C_1}{B_1}, \tag{29}$$

$$v_m^{(i)} = \frac{A}{B_2}\left(\frac{A + \left(2B_2 v_0^{(i)} + 2C_2 - A\right)(m+1)}{2A + \left(4B_2 v_0^{(i)} + 4C_2 - 2A\right)m}\right) - \frac{C_2}{B_2}, \tag{30}$$

when $R = \frac{1}{4}$, that is,

$$u_{2m+i} = \frac{A}{B_1} \frac{\left(B_1 \frac{x_i}{y_{i-1}^{p-1}} + C_1 - \lambda_2 A\right)\lambda_1^{m+1} + \left(A\lambda_1 - B_1 \frac{x_i}{y_{i-1}^{p-1}} - C_1\right)\lambda_2^{m+1}}{\left(B_1 \frac{x_i}{y_{i-1}^{p-1}} + C_1 - \lambda_2 A\right)\lambda_1^{m} + \left(A\lambda_1 - B_1 \frac{x_i}{y_{i-1}^{p-1}} - C_1\right)\lambda_2^{m}} - \frac{C_1}{B_1}, \tag{31}$$

$$v_{2m+i} = \frac{A}{B_2} \frac{\left(B_2 \frac{y_i}{x_{i-1}^{p-1}} + C_2 - \lambda_2 A\right)\lambda_1^{m+1} + \left(A\lambda_1 - B_2 \frac{y_i}{x_{i-1}^{p-1}} - C_2\right)\lambda_2^{m+1}}{\left(B_2 \frac{y_i}{x_{i-1}^{p-1}} + C_2 - A\lambda_2\right)\lambda_1^{m} + \left(A\lambda_1 - B_2 \frac{y_i}{x_{i-1}^{p-1}} - C_2\right)\lambda_2^{m}} - \frac{C_2}{B_2} \tag{32}$$

when $R \neq \frac{1}{4}$, and

$$u_{2m+i} = \frac{A}{B_1}\left(\frac{A + \left(2B_1 \frac{x_i}{y_{i-1}^{p-1}} + 2C_1 - A\right)(m+1)}{2A + \left(4B_1 \frac{x_i}{y_{i-1}^{p-1}} + 4C_1 - 2A\right)m}\right) - \frac{C_1}{B_1}, \tag{33}$$

$$v_{2m+i} = \frac{A}{B_2}\left(\frac{A + \left(2B_2 \frac{y_i}{x_{i-1}^{p-1}} + 2C_2 - A\right)(m+1)}{2A + \left(4B_2 \frac{y_i}{x_{i-1}^{p-1}} + 4C_2 - 2A\right)m}\right) - \frac{C_2}{B_2}, \tag{34}$$

when $R = \frac{1}{4}$, where $A = a\alpha + b\gamma + c\beta + d\delta$, $B_1 = c\alpha + d\gamma$, $C_1 = c\beta + d\delta$, $B_2 = a\gamma + c\delta$, $C_2 = b\gamma + d\delta$ for $i \in \{-1, 0\}$. From (8), we have that

$$x_{2m-1} = u_{2m-1}y_{2m-2}^{p-1} = u_{2m-1}v_{2m-2}^{p-1}x_{2m-3}^{(p-1)^2}, \ m \in \mathbb{N}, \quad (35)$$
$$x_{2m} = u_{2m}y_{2m-1}^{p-1} = u_{2m}v_{2m-1}^{p-1}x_{2m-2}^{(p-1)^2}, \ m \in \mathbb{N}_0,$$

and

$$y_{2m-1} = v_{2m-1}x_{2m-2}^{p-1} = v_{2m-1}u_{2m-2}^{p-1}y_{2m-3}^{(p-1)^2}, \ m \in \mathbb{N}, \quad (36)$$
$$y_{2m} = v_{2m}x_{2m-1}^{p-1} = v_{2m}u_{2m-1}^{p-1}y_{2m-2}^{(p-1)^2}, \ m \in \mathbb{N}_0,$$

from which it follows that

$$x_{2m+i} = x_{-2-i}^{(p-1)^{2(m+1+i)}} \prod_{k=-i}^{m} v_{2k-1+i}^{(p-1)^{(2m+1-2k)}} u_{2k+i}^{(p-1)^{(2m-2k)}} \quad (37)$$

and

$$y_{2m+i} = y_{-2-i}^{(p-1)^{2(m+1+i)}} \prod_{k=-i}^{m} u_{2k-1+i}^{(p-1)^{(2m+1-2k)}} v_{2k+i}^{(p-1)^{(2m-2k)}}, \quad (38)$$

where $m \in \mathbb{N}$ and $i \in \{-1, 0\}$. Using (31)-(34) into (37) and (38), the formulas of solutions of system (4) are obtained.

2.2. Special cases

In this part, we give the formulas of the solution of the system (4) in some special cases concerning the parameters $a, b, c, d, \alpha, \beta, \gamma, \delta$. We have the following results:

- If $(a\alpha + b\gamma)(c\beta + d\delta) - (a\beta + b\delta)(c\alpha + d\gamma) = 0$ and $(a\alpha + c\beta)(b\gamma + d\delta) - (b\alpha + d\beta)(a\gamma + c\delta) = 0$, from (10), (37) and (38), we can write the solution of the system (4) as follows:

$$x_{2m+i} = x_{-2-i}^{(p-1)^{2(m+1+i)}} \prod_{k=-i}^{m} \left(\frac{a\alpha + c\beta}{a\gamma + c\delta}\right)^{(p-1)^{(2m+1-2k)}} \left(\frac{a\alpha + b\gamma}{c\alpha + d\gamma}\right)^{(p-1)^{(2m-2k)}},$$

$$y_{2m+i} = y_{-2-i}^{(p-1)^{2(m+1+i)}} \prod_{k=-i}^{m} \left(\frac{a\alpha + b\gamma}{c\alpha + d\gamma}\right)^{(p-1)^{(2m+1-2k)}} \left(\frac{a\alpha + c\beta}{a\gamma + c\delta}\right)^{(p-1)^{(2m-2k)}},$$

  where $a\gamma + c\delta \neq 0$, $c\alpha + d\gamma \neq 0$, $m \in \mathbb{N}$ and $i \in \{-1, 0\}$.
- If $a\alpha + b\gamma + c\beta + d\delta = 0$, $a\alpha + c\beta + b\gamma + d\delta = 0$, $u_i \neq -\frac{c\beta + d\delta}{c\alpha + d\gamma}$ and $v_i \neq -\frac{b\gamma + d\delta}{a\gamma + c\delta}$ for $i \in \{-1, 0\}$, then the solutions $(u_n)_{n \geq -1}$ and $(v_n)_{n \geq -1}$ are periodic with period four.
- If $c = \gamma = 0$ and $d\delta \neq 0$, then the equations in (10) reduce to second order linear difference equations

$$u_n = \frac{a\alpha}{d\delta}u_{n-2} + \frac{a\beta + b\delta}{d\delta}, \ v_n = \frac{a\alpha}{d\delta}v_{n-2} + \frac{b\alpha + d\beta}{d\delta}, \ n \in \mathbb{N}. \quad (39)$$

from which it follows that

$$u_{2m+i} = \left(\frac{a\alpha}{d\delta}\right)^m u_i + \frac{\alpha\beta + b\delta}{d\delta}\left(\frac{\left(\frac{a\alpha}{d\delta}\right)^k - 1}{\frac{a\alpha}{d\delta} - 1}\right),$$
$$v_{2m+i} = \left(\frac{a\alpha}{d\delta}\right)^m v_i + \frac{b\alpha + d\beta}{d\delta}\left(\frac{\left(\frac{a\alpha}{d\delta}\right)^k - 1}{\frac{a\alpha}{d\delta} - 1}\right), \tag{40}$$

$m \in \mathbb{N}$ and $i \in \{-1, 0\}$, if $\frac{a\alpha}{d\delta} \neq 1$, and

$$u_{2m+i} = u_i + \frac{a\beta + b\delta}{d\delta} m,$$
$$v_{2m+i} = v_i + \frac{b\alpha + d\beta}{d\delta} m, \tag{41}$$

$m \in \mathbb{N}$ and $i \in \{-1, 0\}$, if $\frac{a\alpha}{d\delta} = 1$. Using formulas (40) and (41), for u_{2m+i}, v_{2m+i}, $m \in \mathbb{N}$, $i \in \{-1, 0\}$, in (37) and (38), we can write

$$x_{2m+i} = x_{-2-i}^{(p-1)^{2(m+1+i)}} \times \prod_{k=-i}^{m}\left(\left(\frac{a\alpha}{d\delta}\right)^{k+i}\frac{y_{-1-i}}{x_{-2-i}^{p-1}} + \frac{b\alpha + d\beta}{d\delta}\left(\frac{\left(\frac{a\alpha}{d\delta}\right)^{k+i} - 1}{\frac{a\alpha}{d\delta} - 1}\right)\right)^{(p-1)^{(2m+1-2k)}} \times \left(\left(\frac{a\alpha}{d\delta}\right)^{k}\frac{x_i}{y_{i-1}^{p-1}} + \frac{\alpha\beta + b\delta}{d\delta}\left(\frac{\left(\frac{a\alpha}{d\delta}\right)^{k} - 1}{\frac{a\alpha}{d\delta} - 1}\right)\right)^{(p-1)^{(2m+1-2k)}},$$

$$y_{2m+i} = y_{-2-i}^{(p-1)^{2(m+1+i)}} \times \prod_{k=-i}^{m}\left(\left(\frac{a\alpha}{d\delta}\right)^{k+i}\frac{x_{-1-i}}{y_{-2-i}^{p-1}} + \frac{\alpha\beta + b\delta}{d\delta}\left(\frac{\left(\frac{a\alpha}{d\delta}\right)^{k+i} - 1}{\frac{a\alpha}{d\delta} - 1}\right)\right)^{(p-1)^{(2m+1-2k)}} \times \left(\left(\frac{a\alpha}{d\delta}\right)^{k}\frac{y_i}{x_{i-1}^{p-1}} + \frac{b\alpha + d\beta}{d\delta}\left(\frac{\left(\frac{a\alpha}{d\delta}\right)^{k} - 1}{\frac{a\alpha}{d\delta} - 1}\right)\right)^{(p-1)^{(2m+1-2k)}},$$

$m \in \mathbb{N}$ and $i \in \{-1, 0\}$, if $\frac{a\alpha}{d\delta} \neq 1$, and

$$x_{2m+i} = x_{-2-i}^{(p-1)^{2(m+1+i)}} \prod_{k=-i}^{m}\left(\frac{y_{-1-i}}{x_{-2-i}^{p-1}} + \frac{b\alpha + d\beta}{d\delta}(k+i)\right)^{(p-1)^{(2m+1-2k)}} \times \left(\frac{x_i}{y_{i-1}^{p-1}} + \frac{\alpha\beta + b\delta}{d\delta} k\right)^{(p-1)^{(2m+1-2k)}},$$

$$y_{2m+i} = y_{-2-i}^{(p-1)^{2(m+1+i)}} \prod_{k=-i}^{m}\left(\frac{x_{-1-i}}{y_{-2-i}^{p-1}} + \frac{\alpha\beta + b\delta}{d\delta}(k+i)\right)^{(p-1)^{(2m+1-2k)}}$$

$$\times\left(\frac{y_i}{x_{i-1}^{p-1}}+\frac{b\alpha+d\beta}{d\delta}k\right)^{(p-1)^{(2m+1-2k)}},$$

$m\in\mathbb{N}$ and $i\in\{-1,0\}$, if $\frac{a\alpha}{d\delta}=1$.

3. Forbidden set

In this section, we determine the forbidden set of the initial values for the system (4) via the following theorem.

Theorem 3.1. *The forbidden set of the initial values for the system* (4) *is given by the set*

$$\mathcal{F}=\Bigg\{(x_{-2},x_{-1},y_{-2},y_{-1})\in\mathbb{R}^4: x_{-1}y_{-1}=0 \text{ or } \frac{x_{-1}}{y_{-2}^{p-1}}=(f\circ g)^{-n}\left(-\frac{\delta}{\gamma}\right),$$
$$\text{or },\ \frac{x_{-1}}{y_{-2}^{p-1}}=(f\circ g)^{-n}\left(-\frac{d\delta+b\gamma}{c\delta+a\gamma}\right),\ \text{ or } \frac{y_{-1}}{x_{-2}^{p-1}}=(g\circ f)^{-n}\left(-\frac{d}{c}\right),\ \text{ or },$$
$$\frac{y_{-1}}{x_{-2}^{p-1}}=(g\circ f)^{-n}\left(-\frac{d\delta+c\beta}{d\gamma+c\alpha}\right)\Bigg\}, \tag{42}$$

where

$$(f\circ g)^{-n}(t)=\frac{-A}{\widetilde{B}_1}\frac{\left(\widetilde{B}_1t+\widetilde{C}_1+\lambda_2A\right)\lambda_1^{n+1}-\left(A\lambda_1+\widetilde{B}_1t+\widetilde{C}_1\right)\lambda_2^{n+1}}{\left(\widetilde{B}_1t+\widetilde{C}_1+\lambda_2A\right)\lambda_1^{n}-\left(A\lambda_1+\widetilde{B}_1t+\widetilde{C}_1\right)\lambda_2^{n}}-\frac{\widetilde{C}_1}{\widetilde{B}_1},$$

$$(g\circ f)^{-n}(t)=\frac{-A}{\widetilde{B}_2}\frac{\left(\widetilde{B}_2t+\widetilde{C}_2+\lambda_2A\right)\lambda_1^{n+1}-\left(A\lambda_1+\widetilde{B}_2t+\widetilde{C}_2\right)\lambda_2^{n+1}}{\left(\widetilde{B}_2t+\widetilde{C}_2+A\lambda_2\right)\lambda_1^{n}-\left(A\lambda_1+\widetilde{B}_2t+\widetilde{C}_2\right)\lambda_2^{n}}-\frac{\widetilde{C}_2}{\widetilde{B}_2}$$

when $R\neq\frac{1}{4}$, and

$$(f\circ g)^{-n}(t)=\frac{-A}{\widetilde{B}_1}\left(\frac{-A+\left(2\widetilde{B}_1t+2\widetilde{C}_1+A\right)(n+1)}{-2A+\left(4\widetilde{B}_1t+4\widetilde{C}_1+2A\right)n}\right)-\frac{\widetilde{C}_1}{\widetilde{B}_1},$$

$$(g\circ f)^{-n}(t)=\frac{-A}{\widetilde{B}_2}\left(\frac{-A+\left(2\widetilde{B}_2t+2\widetilde{C}_2+A\right)(n+1)}{-2A+\left(4\widetilde{B}_2t+4\widetilde{C}_2+2A\right)n}\right)-\frac{\widetilde{C}_2}{\widetilde{B}_2}$$

when $R=\frac{1}{4}$, where $A=a\alpha+b\gamma+c\beta+d\delta$, $\widetilde{B}_1=a\gamma+c\delta$, $\widetilde{C}_1=-(a\alpha+c\beta)$, $\widetilde{B}_2=c\alpha+d\gamma$, $\widetilde{C}_2=-(a\alpha+b\gamma)$.

Proof. First, from Lemma 2.1, we conclude that if $x_{-1}y_{-1}=0$, then the value of x_ny_n is undefinable for $n\geq 1$. Second, if $x_ny_n\neq 0$ for $n\geq -2$, then note that the system (4) is undefined, if one of the following conditions

$$cy_{n-1}+dx_{n-2}^{p-1}=0,\quad \gamma x_{n-1}+\delta y_{n-2}^{p-1}=0,\ n\in\mathbb{N}_0 \tag{43}$$

is satisfied. By taking into account the change of variables (8), we can write the corresponding conditions

$$u_{n-1} = -\frac{\delta}{\gamma}, \quad v_{n-1} = -\frac{d}{c}, \ n \in \mathbb{N}_0. \tag{44}$$

Therefore, we can determine the forbidden set of the initial values for the system (4) by using Eq. (9). We know that the statements

$$u_{2n-1} = (f \circ g)^n (u_{-1}), \tag{45}$$

$$u_{2n} = (f \circ g)^n \circ f (u_{-1}), \tag{46}$$

$$v_{2n-1} = (g \circ f)^n (v_{-1}), \tag{47}$$

$$v_{2n} = (g \circ f)^n \circ g (v_{-1}), \tag{48}$$

where

$$f(x) = \frac{ax+b}{cx+d} \quad \text{and} \quad g(x) = \frac{\alpha x + \beta}{\gamma x + \delta},$$

characterize the solutions of Eq. (9). By using the conditions (44) and the statements (45)-(48), we have

$$u_{-1} = (f \circ g)^{-n} \left(-\frac{\delta}{\gamma}\right), \tag{49}$$

$$\begin{aligned} u_{-1} &= f^{-1} \circ (f \circ g)^{-n} \left(-\frac{\delta}{\gamma}\right) = (f \circ g)^{-n} \circ f^{-1} \left(-\frac{\delta}{\gamma}\right) \\ &= (f \circ g)^{-n} \left(-\frac{d\delta + b\gamma}{c\delta + a\gamma}\right), \end{aligned} \tag{50}$$

$$v_{-1} = (g \circ f)^{-n} \left(-\frac{d}{c}\right), \tag{51}$$

$$\begin{aligned} v_{-1} &= g^{-1} \circ (g \circ f)^{-n} \left(-\frac{d}{c}\right) = (g \circ f)^{-n} \circ g^{-1} \left(-\frac{d}{c}\right) \\ &= (g \circ f)^{-n} \left(-\frac{d\delta + c\beta}{d\gamma + c\alpha}\right), \end{aligned} \tag{52}$$

where $c\gamma \neq 0$ and $a + d \neq 0 \neq \alpha + \delta$. Also, let us indicate that the backward solutions of Eq. (9) are the forward solutions of the system

$$t_n = (f \circ g)^{-1} (t_{n-1}), \ \widetilde{t}_n = (g \circ f)^{-1} \left(\widetilde{t}_{n-1}\right), \ n \in \mathbb{N}_0, \tag{53}$$

which corresponds the system

$$t_n = \frac{-(c\beta+d\delta)\, t_{n-2} + a\beta + b\delta}{(c\alpha+d\gamma)\, t_{n-2} - (a\alpha+b\gamma)}, \ \widetilde{t}_n = \frac{-(b\gamma+d\delta)\, \widetilde{t}_{n-2} + b\alpha + d\beta}{(a\gamma+c\delta)\, \widetilde{t}_{n-2} - (a\alpha+c\beta)}, \ n \in \mathbb{N}. \tag{54}$$

By following the procedure used to solve the system (4), one can obtain the solution

$$(55)\qquad t_{2m+i}=\frac{-A}{\widetilde{B}_1}\frac{\left(\widetilde{B}_1t_i+\widetilde{C}_1+\lambda_2A\right)\lambda_1^{m+1}-\left(A\lambda_1+\widetilde{B}_1t_i+\widetilde{C}_1\right)\lambda_2^{m+1}}{\left(\widetilde{B}_1t_i+\widetilde{C}_1+\lambda_2A\right)\lambda_1^{m}-\left(A\lambda_1+\widetilde{B}_1t_i+\widetilde{C}_1\right)\lambda_2^{m}}-\frac{\widetilde{C}_1}{\widetilde{B}_1},$$

$$(56)\quad \widetilde{t}_{2m+i}=\frac{-A}{\widetilde{B}_2}\frac{\left(\widetilde{B}_2\widetilde{t}_i+\widetilde{C}_2+\lambda_2A\right)\lambda_1^{m+1}-\left(A\lambda_1+\widetilde{B}_2\widetilde{t}_i+\widetilde{C}_2\right)\lambda_2^{m+1}}{\left(\widetilde{B}_2\widetilde{t}_i+\widetilde{C}_2+A\lambda_2\right)\lambda_1^{m}-\left(A\lambda_1+\widetilde{B}_2\widetilde{t}_i+\widetilde{C}_2\right)\lambda_2^{m}}-\frac{\widetilde{C}_2}{\widetilde{B}_2}$$

when $R\neq\frac{1}{4}$, and

$$(57)\qquad t_{2m+i}=\frac{-A}{\widetilde{B}_1}\left(\frac{-A+\left(2\widetilde{B}_1t_i+2\widetilde{C}_1+A\right)(m+1)}{-2A+\left(4\widetilde{B}_1t_i+4\widetilde{C}_1+2A\right)m}\right)-\frac{\widetilde{C}_1}{\widetilde{B}_1},$$

$$(58)\qquad t_{2m+i}=\frac{-A}{\widetilde{B}_2}\left(\frac{-A+\left(2\widetilde{B}_2\widetilde{t}_i+2\widetilde{C}_2+A\right)(m+1)}{-2A+\left(4\widetilde{B}_2\widetilde{t}_i+4\widetilde{C}_2+2A\right)m}\right)-\frac{\widetilde{C}_2}{\widetilde{B}_2},$$

when $R=\frac{1}{4}$, for $i\in\{-1,0\}$, where $A=a\alpha+b\gamma+c\beta+d\delta$, $\widetilde{B}_1=a\gamma+c\delta$, $\widetilde{C}_1=-(a\alpha+c\beta)$, $\widetilde{B}_2=c\alpha+d\gamma$, $\widetilde{C}_2=-(a\alpha+b\gamma)$. By applying (49)-(52) and the change of variables (8) to (55)-(58), we obtain the result in (42). □

4. Long-term behavior of solutions in the case $p=2$

In this section, we determine the asymptotic behavior of the solutions of the system (4) when $p=2$. In this case, the system (4) becomes

$$(59)\qquad x_n=\frac{ay_{n-1}^2+bx_{n-2}y_{n-1}}{cy_{n-1}+dx_{n-2}},\ y_n=\frac{\alpha x_{n-1}^2+\beta y_{n-2}x_{n-1}}{\gamma x_{n-1}+\delta y_{n-2}},\ n\in\mathbb{N}_0.$$

The solution of the system (59) is given by

$$x_{2m+i}=x_{-2-i}\prod_{k=-i}^{m}\left(\frac{A}{B_2}\frac{\left(B_2\frac{y_{-1-i}}{x_{-2-i}}+C_2-\lambda_2A\right)\lambda_1^{k+1+i}+\left(A\lambda_1-B_2\frac{y_{-1-i}}{x_{-2-i}}-C_2\right)\lambda_2^{k+1+i}}{\left(B_2\frac{y_{-1-i}}{x_{-2-i}}+C_2-\lambda_2A\right)\lambda_1^{k+i}+\left(A\lambda_1-B_2\frac{y_{-1-i}}{x_{-2-i}}-C_2\right)\lambda_2^{k+i}}-\frac{C_2}{B_2}\right)$$

$$\times\left(\frac{A}{B_1}\frac{\begin{array}{c}\left(B_1\frac{x_i}{y_{i-1}}+C_1-A\lambda_2\right)\lambda_1^{k+1}\\+\left(A\lambda_1-B_1\frac{x_i}{y_{i-1}}-C_1\right)\lambda_2^{k+1}\end{array}}{\begin{array}{c}\left(B_1\frac{x_i}{y_{i-1}}+C_1-A\lambda_2\right)\lambda_1^{k}\\+\left(A\lambda_1-B_1\frac{x_i}{y_{i-1}}-C_1\right)\lambda_2^{k}\end{array}}-\frac{C_1}{B_1}\right), \tag{60}$$

$$y_{2m+i}=y_{-2-i}\prod_{k=-i}^{m}\left(\frac{A}{B_1}\frac{\begin{array}{c}\left(B_1\frac{x_{-1-i}}{y_{-2-i}}+C_1-\lambda_2 A\right)\lambda_1^{k+1+i}\\+\left(A\lambda_1-B_1\frac{x_{-1-i}}{y_{-2-i}}-C_1\right)\lambda_2^{k+1+i}\end{array}}{\begin{array}{c}\left(B_1\frac{x_{-1-i}}{y_{-2-i}}+C_1-\lambda_2 A\right)\lambda_1^{k+i}\\+\left(A\lambda_1-B_1\frac{x_{-1-i}}{y_{-2-i}}-C_1\right)\lambda_2^{k+i}\end{array}}-\frac{C_1}{B_1}\right)$$

$$\times\left(\frac{A}{B_2}\frac{\begin{array}{c}\left(B_2\frac{y_i}{x_{i-1}}+C_2-\lambda_2 A\right)\lambda_1^{k+1}\\+\left(A\lambda_1-B_2\frac{y_i}{x_{i-1}}-C_2\right)\lambda_2^{k+1}\end{array}}{\begin{array}{c}\left(B_2\frac{y_i}{x_{i-1}}+C_2-\lambda_2 A\right)\lambda_1^{k}\\+\left(A\lambda_1-B_2\frac{y_i}{x_{i-1}}-C_2\right)\lambda_2^{k}\end{array}}-\frac{C_2}{B_2}\right) \tag{61}$$

when $R\neq\frac{1}{4}$, and

$$\begin{aligned}x_{2m+i}&=x_{-2-i}\\&\times\prod_{k=-i}^{m}\left(\frac{A}{B_2}\frac{A+\left(2B_2\frac{y_{-1-i}}{x_{-2-i}}+2C_2-A\right)(k+1+i)}{2A+\left(4B_2\frac{y_{-1-i}}{x_{-2-i}}+4C_2-2A\right)(k+i)}-\frac{C_2}{B_2}\right)\\&\times\left(\frac{A}{B_1}\frac{A+\left(2B_1(i)\frac{x_i}{y_{i-1}}+2C_1-A\right)(k+1)}{2A+\left(4B_1\frac{x_i}{y_{i-1}}+4C_1-2A\right)k}-\frac{C_1}{B_1}\right),\end{aligned} \tag{62}$$

$$\begin{aligned}y_{2m+i}&=y_{-2-i}\\&\times\prod_{k=-i}^{m}\left(\frac{A}{B_1}\frac{A+\left(2B_1\frac{x_{-1-i}}{y_{-2-i}}+2C_1-A\right)(k+1+i)}{2A+\left(4B_1\frac{x_{-1-i}}{y_{-2-i}}+4C_1-2A\right)(k+i)}-\frac{C_1}{B_1}\right)\\&\times\left(\frac{A}{B_2}\frac{A+\left(2B_2\frac{y_i}{x_{i-1}}+2C_2-A\right)(k+1)}{2A+\left(4B_2\frac{y_i}{x_{i-1}}+4C_2-2A\right)k}-\frac{C_2}{B_2}\right)\end{aligned} \tag{63}$$

when $R = \frac{1}{4}$, where $A = a\alpha + b\gamma + c\beta + d\delta$, $B_1 = c\alpha + d\gamma$, $C_1 = c\beta + d\delta$, $B_2 = a\gamma + c\delta$, $C_2 = b\gamma + d\delta$ for $i \in \{-1, 0\}$ and $m \in \mathbb{N}_0$.

Theorem 4.1. *Assume that $\{(x_n, y_n)\}_{n\geq -2}$ is a well-defined solution of the system (4), $R = \frac{(bc-ad)(\beta\gamma-\alpha\delta)}{(a\alpha+b\gamma+c\beta+d\delta)^2} \neq \frac{1}{4}$, $\frac{x_i}{y_{i-1}} \neq \frac{\lambda_j A - C_1}{B_1}$, $\frac{y_i}{x_{i-1}} \neq \frac{\lambda_j A - C_2}{B_2}$, $L_j := \frac{\lambda_j A - C_1}{B_1}$ and $M_j := \frac{\lambda_j A - C_2}{B_2}$ for $i \in \{-1, 0\}$ and $j \in \{1, 2\}$. Then the following statements are true.*

(a) *If $|\lambda_1| > |\lambda_2|$ and $|M_1 L_1| < 1$, then $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$.*
(b) *If $|\lambda_1| > |\lambda_2|$ and $|M_1 L_1| > 1$, then $|x_n| \to \infty$ and $|y_n| \to \infty$ as $n \to \infty$.*
(c) *If $|\lambda_1| > |\lambda_2|$ and $M_1 L_1 = 1$, then $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ are convergent.*
(d) *If $|\lambda_1| > |\lambda_2|$ and $M_1 L_1 = -1$, then $(x_{2n+i})_{n\geq -1}$ and $(y_{2n+i})_{n\geq -1}$, for $i \in \{-1, 0\}$, are convergent.*
(e) *If $|\lambda_2| > |\lambda_1|$ and $|M_2 L_2| < 1$, then $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$.*
(f) *If $|\lambda_2| > |\lambda_1|$ and $|M_2 L_2| > 1$, then $|x_n| \to \infty$ and $|y_n| \to \infty$ as $n \to \infty$.*
(g) *If $|\lambda_2| > |\lambda_1|$ and $M_2 L_2 = 1$, then $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ are convergent.*
(h) *If $|\lambda_2| > |\lambda_1|$ and $M_2 L_2 = -1$, then $(x_{2n+i})_{n\geq -1}$ and $(y_{2n+i})_{n\geq -1}$, for $i \in \{-1, 0\}$, are convergent.*

Proof. Let

$$a^i_{m_1} = \left(\frac{A}{B_2} \frac{\left(B_2 \frac{y_{-1-i}}{x_{-2-i}} + C_2 - \lambda_2 A\right) \lambda_1^{m_1+1+i} + \left(A\lambda_1 - B_2 \frac{y_{-1-i}}{x_{-2-i}} - C_2\right) \lambda_2^{m_1+1+i}}{\left(B_2 \frac{y_{-1-i}}{x_{-2-i}} + C_2 - \lambda_2 A\right) \lambda_1^{m_1+i} + \left(A\lambda_1 - B_2 \frac{y_{-1-i}}{x_{-2-i}} - C_2\right) \lambda_2^{m_1+i}} - \frac{C_2}{B_2} \right)$$

$$\times \left(\frac{A}{B_1} \frac{\left(B_1 \frac{x_i}{y_{i-1}} + C_1 - A\lambda_2\right) \lambda_1^{m_1+1} + \left(A\lambda_1 - B_1 \frac{x_i}{y_{i-1}} - C_1\right) \lambda_2^{m_1+1}}{\left(B_1 \frac{x_i}{y_{i-1}} + C_1 - A\lambda_2\right) \lambda_1^{m_1} + \left(A\lambda_1 - B_1 \frac{x_i}{y_{i-1}} - C_1\right) \lambda_2^{m_1}} - \frac{C_1}{B_1} \right) \tag{64}$$

and

$$\widehat{a}^{i}_{m_1} = \left(\frac{A}{B_1} \frac{\left(B_1 \frac{x_{-1-i}}{y_{-2-i}} + C_1 - \lambda_2 A\right) \lambda_1^{m_1+1+i} + \left(A\lambda_1 - B_1 \frac{x_{-1-i}}{y_{-2-i}} - C_1\right) \lambda_2^{m_1+1+i}}{\left(B_1 \frac{x_{-1-i}}{y_{-2-i}} + C_1 - \lambda_2 A\right) \lambda_1^{m_1+i} + \left(A\lambda_1 - B_1 \frac{x_{-1-i}}{y_{-2-i}} - C_1\right) \lambda_2^{m_1+i}} - \frac{C_1}{B_1} \right)$$

$$\times \left(\frac{A}{B_2} \frac{\left(B_2 \frac{y_i}{x_{i-1}} + C_2 - \lambda_2 A\right) \lambda_1^{m_1+1} + \left(A\lambda_1 - B_2 \frac{y_i}{x_{i-1}} - C_2\right) \lambda_2^{m_1+1}}{\left(B_2 \frac{y_i}{x_{i-1}} + C_2 - \lambda_2 A\right) \lambda_1^{m_1} + \left(A\lambda_1 - B_2 \frac{y_i}{x_{i-1}} - C_2\right) \lambda_2^{m_1}} - \frac{C_2}{B_2} \right) \tag{65}$$

for $m_1 \in \mathbb{N}_0$ and $i \in \{-1, 0\}$. Then if $|\lambda_1| > |\lambda_2|$, we get that for each $i \in \{-1, 0\}$

$$\lim_{m_1 \to \infty} a^{i}_{m_1} = \lim_{m_1 \to \infty} \widehat{a}^{i}_{m_1} = \left(\frac{A\lambda_1 - C_1}{B_1} \right) \left(\frac{A\lambda_1 - C_2}{B_2} \right). \tag{66}$$

From (60), (61) and (66), the results follow from the assumptions in (a) and (b). For each $i \in \{-1, 0\}$ and a sufficiently large m_1 we can write

$$a^{i}_{m_1} = \left(-\frac{C_2}{B_2} + \frac{A}{B_2} \frac{\lambda_1 + \lambda_1 \left(\frac{A\lambda_1 - B_2 \frac{y_{-1-i}}{x_{-2-i}} - C_2}{B_2 \frac{y_{-1-i}}{x_{-2-i}} + C_2 - \lambda_2 A} \right) \left(\frac{\lambda_2}{\lambda_1} \right)^{m_1+1+i}}{1 + \left(\frac{A\lambda_1 - B_2 \frac{y_{-1-i}}{x_{-2-i}} - C_2}{B_2 \frac{y_{-1-i}}{x_{-2-i}} + C_2 - \lambda_2 A} \right) \left(\frac{\lambda_2}{\lambda_1} \right)^{m_1+i}} \right)$$

$$\times \left(-\frac{C_1}{B_1} + \frac{A}{B_1} \frac{\lambda_1 + \lambda_1 \left(\frac{A\lambda_1 - B_1 \frac{x_i}{y_{i-1}} - C_1}{B_1 \frac{x_i}{y_{i-1}} + C_1 - \lambda_2 A} \right) \left(\frac{\lambda_2}{\lambda_1} \right)^{m_1+1}}{1 + \left(\frac{A\lambda_1 - B_1 \frac{x_i}{y_{i-1}} - C_1}{B_1 \frac{x_i}{y_{i-1}} + C_2 - \lambda_2 A} \right) \left(\frac{\lambda_2}{\lambda_1} \right)^{m_1}} \right) \tag{67}$$

$$= \left(-\frac{C_2}{B_2} + \frac{A\lambda_1}{B_2} + \frac{A}{B_2} \left(\frac{A\lambda_1 - B_2 \frac{y_{-1-i}}{x_{-2-i}} - C_2}{B_2 \frac{y_{-1-i}}{x_{-2-i}} + C_2 - \lambda_2 A} \right) (\lambda_2 - \lambda_1) \left(\frac{\lambda_2}{\lambda_1} \right)^{m_1+i} + \mathcal{O} \left(\frac{\lambda_2}{\lambda_1} \right)^{2m_1} \right)$$

$$\times \left(-\frac{C_1}{B_1} + \frac{A\lambda_1}{B_1} + \frac{A}{B_1} \left(\frac{A\lambda_1 - B_1 \frac{x_i}{y_{i-1}} - C_1}{B_1 \frac{x_i}{y_{i-1}} + C_1 - \lambda_2 A} \right) (\lambda_2 - \lambda_1) \left(\frac{\lambda_2}{\lambda_1} \right)^{m_1}$$

$$
\begin{aligned}
&+\mathcal{O}\left(\frac{\lambda_2}{\lambda_1}\right)^{2m_1}\Bigg)\\
&= M_1 L_1 + \left(\frac{L_1}{B_2}\left(\frac{\lambda_2}{\lambda_1}\right)^i \frac{A\lambda_1 - B_2\frac{y_{-1-i}}{x_{-2-i}} - C_2}{B_2\frac{y_{-1-i}}{x_{-2-i}} + C_2 - \lambda_2 A} + \frac{M_1}{B_1}\frac{A\lambda_1 - B_1\frac{x_i}{y_{i-1}} - C_1}{B_1\frac{x_i}{y_{i-1}} + C_1 - \lambda_2 A}\right)\\
&\quad \times A\left(\lambda_2 - \lambda_1\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{m_1} + \mathcal{O}\left(\frac{\lambda_2}{\lambda_1}\right)^{2m_1}
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{a}^i_{m_1} &= \left(-\frac{C_1}{B_1} + \frac{A}{B_1}\frac{\lambda_1 + \lambda_1\left(\frac{A\lambda_1 - B_1\frac{x_{-1-i}}{y_{-2-i}} - C_1}{B_1\frac{x_{-1-i}}{y_{-2-i}} + C_1 - \lambda_2 A}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{m_1+1+i}}{1 + \left(\frac{A\lambda_1 - B_1\frac{x_{-1-i}}{y_{-2-i}} - C_1}{B_1\frac{x_{-1-i}}{y_{-2-i}} + C_2 - \lambda_2 A}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{m_1+i}}\right)\\
&\quad \times \left(-\frac{C_2}{B_2} + \frac{A}{B_2}\frac{\lambda_1 + \lambda_1\left(\frac{A\lambda_1 - B_2\frac{y_i}{x_{i-1}} - C_2}{B_2\frac{y_i}{x_{i-1}} + C_2 - \lambda_2 A}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{m_1+1}}{1 + \left(\frac{A\lambda_1 - B_2\frac{y_i}{x_{i-1}} - C_2}{B_2\frac{y_i}{x_{i-1}} + C_2 - \lambda_2 A}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{m_1}}\right)\\
&= \left(-\frac{C_1}{B_1} + \frac{A\lambda_1}{B_2} + \frac{A}{B_1}\left(\frac{A\lambda_1 - B_1\frac{x_{-1-i}}{y_{-2-i}} - C_1}{B_1\frac{x_{-1-i}}{y_{-2-i}} + C_1 - \lambda_2 A}\right)\left(\lambda_2 - \lambda_1\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{m_1+i}\right.\\
&\quad \left. + \mathcal{O}\left(\frac{\lambda_2}{\lambda_1}\right)^{2m_1}\right)\\
&\quad \times \left(-\frac{C_2}{B_2} + \frac{A\lambda_2}{B_2} + \frac{A}{B_2}\left(\frac{A\lambda_1 - B_2\frac{y_i}{x_{i-1}} - C_2}{B_2\frac{y_i}{x_{i-1}} + C_2 - \lambda_2 A}\right)\left(\lambda_2 - \lambda_1\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{m_1}\right.\\
&\quad \left. + \mathcal{O}\left(\frac{\lambda_2}{\lambda_1}\right)^{2m_1}\right)\\
&= L_1 M_1 + \left(\frac{M_1}{B_1}\left(\frac{\lambda_2}{\lambda_1}\right)^i \frac{A\lambda_1 - B_1\frac{x_{-1-i}}{y_{-2-i}} - C_1}{B_1\frac{x_{-1-i}}{y_{-2-i}} + C_1 - \lambda_2 A} + \frac{L_1}{B_2}\frac{A\lambda_1 - B_2\frac{y_i}{x_{i-1}} - C_2}{B_2\frac{y_i}{x_{i-1}} + C_2 - \lambda_2 A}\right)\\
&\quad \times A\left(\lambda_2 - \lambda_1\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{m_1} + \mathcal{O}\left(\frac{\lambda_2}{\lambda_1}\right)^{2m_1}.
\end{aligned} \tag{68}
$$

From (60), (61), (67) and (68), the results in (c) and (d) can be seen easily. The proofs of the statements (e)-(h) are similar with those of (a)-(d) and thus they are omitted. □

Theorem 4.2. *Assume that* $\{(x_n, y_n)\}_{n\geq -2}$ *is a well-defined solution of the system* (4), $R = \frac{(bc-ad)(\beta\gamma-\alpha\delta)}{(a\alpha+b\gamma+c\beta+d\delta)^2} = \frac{1}{4}$, $x_{-2-i}, y_{-2-i} \neq 0$ *for* $i \in \{-1, 0\}$, $A = a\alpha + b\gamma + c\beta + d\delta$, $B_1 = c\alpha + d\gamma$, $C_1 = c\beta + d\delta$, $B_2 = a\gamma + c\delta$ *and* $C_2 = b\gamma + d\delta$. *Then the following statements are true.*

(a) *If* $\left|\frac{(A-2C_1)(A-2C_2)}{4B_1B_2}\right| < 1$, *then* $x_n \to 0$ *and* $y_n \to 0$ *as* $n \to \infty$.

(b) *If* $|\frac{(A-2C_1)(A-2C_2)}{4B_1B_2}| > 1$*, then* $|x_n| \to \infty$ *and* $|y_n| \to \infty$ *as* $n \to \infty$*.*
(c) *If* $|\frac{(A-2C_1)(A-2C_2)}{4B_1B_2}| = 1$ *and* $\frac{(A-2C_1)(A-2C_2)}{2A(A-C_1-C_2)} > 0$*, then* $|x_n| \to \infty$ *and* $|y_n| \to \infty$ *as* $n \to \infty$*.*
(d) *If* $|\frac{(A-2C_1)(A-2C_2)}{4B_1B_2}| = 1$ *and* $\frac{(A-2C_1)(A-2C_2)}{2A(A-C_1-C_2)} < 0$*, then* $x_n \to 0$ *and* $y_n \to 0$ *as* $n \to \infty$*.*

Proof. If $R = \frac{(bc-ad)(\beta\gamma-\alpha\delta)}{(a\alpha+b\gamma+c\beta+d\delta)^2} = \frac{1}{4}$, then we get $\lambda_1 = \lambda_2 = \frac{1}{2}$. Let

$$b^i_{m_1} := \left(\frac{A}{B_2}\frac{A+\left(2B_2\frac{y_{-1-i}}{x_{-2-i}}+2C_2-A\right)(m_1+1+i)}{2A+\left(4B_2\frac{y_{-1-i}}{x_{-2-i}}+4C_2-2A\right)(m_1+i)}-\frac{C_2}{B_2}\right) \times \left(\frac{A}{B_1}\frac{A+\left(2B_1\frac{x_i}{y_{i-1}}+2C_1-A\right)(m_1+1)}{2A+\left(4B_1\frac{x_i}{y_{i-1}}+4C_1-2A\right)m_1}-\frac{C_1}{B_1}\right) \tag{69}$$

and

$$\widehat{b}^i_{m_1} := \left(\frac{A}{B_1}\frac{A+\left(2B_1\frac{x_{-1-i}}{y_{-2-i}}+2C_1-A\right)(m_1+1+i)}{2A+\left(4B_1\frac{x_{-1-i}}{y_{-2-i}}+4C_1-2A\right)(m_1+i)}-\frac{C_1}{B_1}\right) \times \left(\frac{A}{B_2}\frac{A+\left(2B_2\frac{y_i}{x_{i-1}}+2C_2-A\right)(m_1+1)}{2A+\left(4B_2\frac{y_i}{x_{i-1}}+4C_2-2A\right)m_1}-\frac{C_2}{B_2}\right) \tag{70}$$

for every $m \in \mathbb{N}_0$ and $i \in \{-1,0\}$. If at least one of coefficients of m_1 is different from 0, then we have

$$\lim_{m_1\to\infty} b^i_{m_1} = \frac{(A-2C_1)(A-2C_2)}{4B_1B_2} = \lim_{m_1\to\infty} \widehat{b}^i_{m_1} \tag{71}$$

for each $i \in \{-1,0\}$. Otherwise, when $\frac{x_i}{y_{i-1}} = \frac{A-2C_1}{2B_1}$ and $\frac{y_{-1-i}}{x_{-2-i}} = \frac{A-2C_2}{2B_2}$ for $i \in \{-1,0\}$, directly we get equivalent in (71). From (64), (65) and (71), the results follow from the assumptions in (a) and (b). Now we consider the other cases. For each $i \in \{-1,0\}$ and sufficiently large m_1, we obtain

$$\begin{aligned} b^i_{m_1} = \widehat{b}^i_{m_1} &= \left(-\frac{C_2}{B_2}+\frac{A}{B_2}\left(\frac{1}{2}+\frac{1}{2m_1}+\mathcal{O}\left(\frac{1}{m_1^2}\right)\right)\right) \\ &\quad\times\left(-\frac{C_1}{B_1}+\frac{A}{B_1}\left(\frac{1}{2}+\frac{1}{2m_1}+\mathcal{O}\left(\frac{1}{m_1^2}\right)\right)\right) \\ &= \left(\frac{A-2C_2}{2B_2}+\frac{A}{2B_2m_1}+\mathcal{O}\left(\frac{1}{m_1^2}\right)\right) \\ &\quad\times\left(\frac{A-2C_1}{2B_1}+\frac{A}{2B_1m_1}+\mathcal{O}\left(\frac{1}{m_1^2}\right)\right) \end{aligned}$$

$$= \frac{(A-2C_1)(A-2C_2)}{4B_1B_2}\left(1+\frac{\frac{2A(A-C_1-C_2)}{(A-2C_1)(A-2C_2)}}{m_1}+\mathcal{O}\left(\frac{1}{m_1^2}\right)\right)$$

$$= \pm\left(1+\frac{1}{\frac{(A-2C_1)(A-2C_2)}{2A(A-C_1-C_2)}m_1}+\mathcal{O}\left(\frac{1}{m_1^2}\right)\right)$$

$$= \pm\exp\left(\frac{1}{\frac{(A-2C_1)(A-2C_2)}{2A(A-C_1-C_2)}m_1}+\mathcal{O}\left(\frac{1}{m_1^2}\right)\right). \tag{72}$$

From (69), (70) and (72) by using the fact that $\Sigma_{j_1=1}^{m_1}(1/j_1)\to\infty$ as $m_1\to\infty$, then the statements are easily obtained. □

5. Conclusion

We mainly conclude from this study that the system (4) can be solved in closed form by means of Eq. (1) of Riccati type. Also, we investigated some special cases of the system (4) corresponding to necessary restrictions of the change of variables used in solving of Eq. (1). Moreover, we studied existence and long-term behavior in the case $p=2$ of the solutions. Since the present system is a two dimensional natural extension to Eq. (2), we extended the results in [15].

References

[1] L. Brand, *Classroom notes: a sequence defined by a difference equation*, Amer. Math. Monthly **62** (1955), no. 7, 489–492. https://doi.org/10.2307/2307362

[2] M. Dehghan, R. Mazrooei-Sebdani, and H. Sedaghat, *Global behaviour of the Riccati difference equation of order two*, J. Difference Equ. Appl. **17** (2011), no. 4, 467–477. https://doi.org/10.1080/10236190903049017

[3] I. Dekkar, N. Touafek, and Y. Yazlik, *Global stability of a third-order nonlinear system of difference equations with period-two coefficients*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **111** (2017), no. 2, 325–347. https://doi.org/10.1007/s13398-016-0297-z

[4] S. N. Elaydi, *An Introduction to Difference Equations*, second edition, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1999. https://doi.org/10.1007/978-1-4757-3110-1

[5] M. M. El-Dessoky and E. M. Elsayed, *On the solutions and periodic nature of some systems of rational difference equations*, J. Comput. Anal. Appl. **18** (2015), no. 2, 206–218. http://doi.org/10.1166/jctn.2015.4263

[6] E. M. Elsayed, *On the solutions and periodic nature of some systems of difference equations*, Int. J. Biomath. **7** (2014), no. 6, 1450067, 26 pp. https://doi.org/10.1142/S1793524514500673

[7] N. Haddad, N. Touafek, and J. F. T. Rabago, *Solution form of a higher-order system of difference equations and dynamical behavior of its special case*, Math. Methods Appl. Sci. **40** (2017), no. 10, 3599–3607. https://doi.org/10.1002/mma.4248

[8] ———, *Well-defined solutions of a system of difference equations*, J. Appl. Math. Comput. **56** (2018), no. 1-2, 439–458. https://doi.org/10.1007/s12190-017-1081-8

[9] Y. Halim and M. Bayram, *On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences*, Math. Methods Appl. Sci. **39** (2016), no. 11, 2974–2982. https://doi.org/10.1002/mma.3745

[10] Y. Halim, N. Touafek, and Y. Yazlik, *Dynamic behavior of a second-order nonlinear rational difference equation*, Turkish J. Math. **39** (2015), no. 6, 1004–1018. https://doi.org/10.3906/mat-1503-80

[11] T. F. Ibrahim and N. Touafek, *On a third order rational difference equation with variable coefficients*, DCDIS Series B: Applications & Algorithms **20**, (2013), no. 2, 251–264.

[12] E. A. Grove, Y. Kostrov, G. Ladas, and S. W. Schultz, *Riccati difference equations with real period-2 coefficients*, Comm. Appl. Nonlinear Anal. **14** (2007), no. 2, 33–56.

[13] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations*, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[14] H. Matsunaga and R. Suzuki, *Classification of global behavior of a system of rational difference equations*, Appl. Math. Lett. **85** (2018), 57–63. https://doi.org/10.1016/j.aml.2018.05.020

[15] L. C. McGrath and C. Teixeira, *Existence and behavior of solutions of the rational equation $x_{n+1} = (ax_{n-1} + bx_n)/(cx_{n-1} + dx_n)x_n$, $n = 0, 1, 2, \ldots$*, Rocky Mountain J. Math. **36** (2006), no. 2, 649–674. https://doi.org/10.1216/rmjm/1181069472

[16] S. Reich and A. J. Zaslavski, *Asymptotic behavior of a dynamical system on a metric space*, J. Nonlinear Variational Anal. **3** (2019), 79–85.

[17] H. Sedaghat, *Global behaviours of rational difference equations of orders two and three with quadratic terms*, J. Difference Equ. Appl. **15** (2009), no. 3, 215–224. https://doi.org/10.1080/10236190802054126

[18] S. Selvarangam, S. Geetha, and E. Thandapani, *Existence of nonoscillatory solutions to second order neutral type difference equations with mixed arguments*, Int. J. Difference Equ. **13** (2018), no. 1, 55–69.

[19] S. Stević, *On a system of difference equations with period two coefficients*, Applied Mathematics and Computation **218** (2011), no. 8, 4317–4324. https://doi.org/10.1016/j.amc.2011.10.005

[20] ———, *On some solvable systems of difference equations*, Appl. Math. Comput. **218** (2012), no. 9, 5010–5018. https://doi.org/10.1016/j.amc.2011.10.068

[21] ———, *Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences*, Electronic Journal of Qualitative Theory of Differential Equations **2014** (2014), no. 67, 1–15.

[22] S. Stević, M. A. Alghamdi, N. Shahzad, and D. A. Maturi, *On a class of solvable difference equations*, Abstr. Appl. Anal. **2013** (2013), Art. ID 157943, 7 pp. https://doi.org/10.1155/2013/157943

[23] D. T. Tollu, Y. Yazlik, and N. Taskara, *On the solutions of two special types of Riccati difference equation via Fibonacci numbers*, Adv. Difference Equ. **2013** (2013), 174, 7 pp. https://doi.org/10.1186/1687-1847-2013-174

[24] ———, *On fourteen solvable systems of difference equations*, Appl. Math. Comput. **233** (2014), 310–319. https://doi.org/10.1016/j.amc.2014.02.001

[25] ———, *On a solvable nonlinear difference equation of higher order*, Turkish J. Math. **42** (2018), no. 4, 1765–1778. https://doi.org/10.3906/mat-1705-33

[26] Q. Wang and Q. Zhang, *Dynamics of a higher-order rational difference equation*, J. Appl. Anal. Comput. **7** (2017), no. 2, 770–787. http://doi.org/10.11948/2017048

[27] Y. Yazlık, *On the solutions and behavior of rational difference equations*, J. Comput. Anal. Appl. **17** (2014), no. 3, 584–594.

[28] Y. Yazlık, D. T. Tollu, and N. Taşkara, *On the solutions of a three-dimensional system of difference equations*, Kuwait J. Sci. **43** (2016), no. 1, 95–111.

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