

Dynamical analysis of a discrete conformable fractional order bacteria population model in a microcosm

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ABSTRACT

In this paper, conformable fractional order differential equations with piecewise constant arguments are used for a modeling population density of a bacteria species in a microcosm. The discretization process of a differential equation with piecewise constant arguments gives us two dimensional discrete dynamical system in the interval $t \in [nh, (n + 1)h)$. By using the center manifold theorem and the bifurcation theory, it is shown that the discrete dynamical system undergoes flip and Neimark–Sacker bifurcation depending on the parameter r . It is observed that the model describes some biological phenomena for a bacteria population such as homogeneous bacteria distributions and inhomogeneous spatial population distributions. In addition, the effect of fractional order derivative on dynamic behavior of the system is investigated. Finally, all theoretical results are supported by numerical simulations.

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1. Introduction

Micro- and mesocosms are experimental ecosystems of any habitat type, which contain more than one species that can investigate various ecological endpoints. They provide a bridge between laboratory and field studies, preserving some of the natural conversion of field studies, with a greater control of conditions and changeables, as well as reproduction, and are therefore an extremely useful tool to investigate how ecosystems answer to stress. Therefore microcosms have been used as tools [1–7] for studying population dynamics such as bacteria growth. In study [5], Gause created a microcosm of a nutritive decoction of bacteria and studied the variations in the populations over a few days. It is observed that experimental and theoretical results are consistent with each other. Grigoryan et al. [6] have examined the diversity and sulfide-producing activity of microorganisms in microcosms containing commercial clay products. In study [7], Multi-species microcosms have been used to investigate effects of ionizing radiation in a model freshwater ecosystem, including endpoints at both structural and functional levels and ecological interactions. In study [8], substrate supplied to the microhabitats facilitates bacterial growth, and microbial cells can migrate between neighboring microhabitats due to random motility, chemotaxis towards substrate and self-attraction.

The mathematical approximation for population growth in some biological situations includes nonlinear differential equations. It is generally known that a differential equation model is preferred for the overlapping generation of a single species [9,10]. If there is only single species that does not overlap, then it is appropriate to build a model with the difference equation [11]. In recent years, there is a significant increase in the use of fractional-order derivatives into

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many disciplines such as mechanics, chemistry, biology, mathematics and physics [12–15]. The fractional derivative gives an excellent instrument for the description of memory and hereditary properties of various materials and processes. It has been exposed that, the fractional derivative is more suitable for modeling real world problem than ordinary derivative of integer order [16–20].

There are many definitions of the fractional derivative and the most common known are Riemann–Liouville and Caputo derivative. However, Caputo and Riemann–Liouville also have a big problem that, their kernel although nonlocal but is singular. This weakness has effect when modeling real world problem [21]. Therefore, some researchers have introduced the concept of non-local derivative. In [22], Khalil presented a new definition of derivative prominently compatible with the classical derivative. This operator is called conformable derivative and satisfy some conventional properties such as chain rule and have properties similar to the ordinary Newton's calculus. Moreover, biological and physical applications of conformable fractional derivative can be found in [23–25]. According to this definition, the left conformable fractional derivative starting from a of the function $f : [a, \infty) \rightarrow \infty$ of order $0 < \alpha \leq 1$ is defined by

$$(T_{\alpha}^a f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon(t - a)^{1-\alpha}) - f(t)}{\epsilon} \quad (1)$$

and the right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at b of f is defined by

$$({}^b T_{\alpha} f)(t) = - \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon(b - t)^{1-\alpha}) - f(t)}{\epsilon}. \quad (2)$$

Note that if f is differentiable then

$$(T_{\alpha}^a f)(t) = (t - a)^{1-\alpha} f'(t) \quad , \quad ({}^b T_{\alpha} f)(t) = -(b - t)^{1-\alpha} f'(t). \quad (3)$$

In dynamical system, bifurcation and chaos have become an important topic and appear naturally in several important biological models [10,26–31]. In study [10], May has already pointed out that a simple deterministic model can exhibit chaotic behaviors, and the importance of chaos in ecology. Chaotic behavior can be observed as a result of flip and hopf bifurcation that occur with the presence of periodic or quasi-periodic solutions and in this case the dynamic behavior of the system is very sensitive to the initial conditions. As a generalization of the integer-order systems, the fractional-order nonlinear dynamic systems display numerous dynamic behaviors as in integer-order systems, such as attractor, bifurcation, and chaos. Studies have shown that fractional order nonlinear systems exhibit more complex dynamic behavior than the integer-order systems because of its memory effect. Therefore, the study of fractional chaotic systems has become increasingly interesting and numerous fractional nonlinear systems with chaotic behaviors have been found in [32–35].

Another approach taken into account in modeling biological systems is using differential equations with piecewise constant arguments. These equations describe hybrid dynamical systems and combine properties of both differential and difference equations and have applications in widely expanded areas such as biomedicine, chemistry, mechanical engineering, physics and population dynamics [26,27]. Ozturk et al. [4] modeled a population density of a bacteria population in a microcosm such as

$$\frac{dx(t)}{dt} = rx(t)(1 - \alpha x(t) - \beta_0 x([t]) - \beta_1 x([t - 1])) \quad (4)$$

where $t \geq 0$ and $[t]$ denotes the integer part of $t \in [0, \infty)$. $x(t)$ represent green algae (Chlorella) and rotifers (Lepadella) bacteria population in a subculture solution (10 ml) and the initial value of the bacteria population density is nearly 10^4 (cells/ml) and the equilibrium point is approximately $\bar{x} = 5 \times 10^5$ and also the upper bound of the bacteria population density is 7×10^7 . The parameter r is the population growth rate of the bacteria population, α , β_0 and β_1 are coefficients that each represents the irregular environmental carrying capacity for a logistic population model.

The aim of this study is to investigate dynamical behavior of conformable fractional order form of the model (4) that is given as follows;

$$T_q x(t) = rx(t)(1 - \alpha x(t) - \beta_0 x(\lfloor \frac{t}{h} \rfloor h) - \beta_1 x(\lfloor \frac{t-h}{h} \rfloor h)). \quad (5)$$

where $[t]$ denotes the integer part of $t \in [0, \infty)$ and h is discretization parameter.

In the literature, there are limited numbers of studies about modeling population dynamics by using conformable fractional order differential equations [23–25]. In our study, we use conformable fractional order which capture local memory effect for a population but this event cannot be reflected with ordinary differential equations. Biologically, memory effect means that the growth rates of populations not only depend on the current conditions, but also depend on all previous conditions. In view of this, model (5) is a more realistic than the model (4) formed by integer-order differential equations. In addition, we also deal with relatively new discretization process based on the piecewise constant functions for conformable fractional order differential equations [25]. The use of piecewise constant functions allows us to convert differential equation (5) to difference equations. Hence, flip bifurcation, Neimark–Sacker bifurcation and chaotic dynamics for a bacteria population will be easily examined through this discrete system.

2. Discretization process

In this section, we will discretize the model (5) based on approximation given in [25]. From the left conformable fractional derivative, we get

$$(t - nh)^{1-q} \frac{dx(t)}{dt} = rx(t)(1 - \alpha x(t) - \beta_0 x(\lfloor \frac{nh}{h} \rfloor h) - \beta_1 x(\lfloor \frac{nh-h}{h} \rfloor h)). \quad (6)$$

We can rewrite Eq. (6) as following form,

$$-\frac{x'(t)}{x^2(t)} - \frac{r\beta_0 x(nh) + r\beta_1 x(nh-h) - r}{(t-nh)^{1-q}} \frac{1}{x(t)} = \frac{\alpha r}{(t-nh)^{1-q}}. \quad (7)$$

Solving this equation with respect to $t \in [nh, t)$, we obtain

$$x(t) = \frac{(1 - \beta_0 x(nh) - \beta_1 x((n-1)h))x(nh)}{(1 - \beta_0 x(nh) - \beta_1 x((n-1)h) - \alpha x(nh))e^{-r(1 - \beta_0 x(nh) - \beta_1 x((n-1)h))\frac{(t-nh)^q}{q}} + \alpha x(nh)}. \quad (8)$$

Let $t \rightarrow (n+1)h$, then we have the following difference equation

$$x((n+1)h) = \frac{(1 - \beta_0 x(nh) - \beta_1 x((n-1)h))x(nh)}{(1 - \beta_0 x(nh) - \beta_1 x((n-1)h) - \alpha x(nh))e^{-r(1 - \beta_0 x(nh) - \beta_1 x((n-1)h))\frac{h^q}{q}} + \alpha x(nh)}. \quad (9)$$

Finally, adjusting difference equation notation and replacing $x(nh)$ and $x((n-1)h)$ by $x(n)$ and $x(n-1)$ yields

$$x(n+1) = \frac{(1 - \beta_0 x(n) - \beta_1 x(n-1))x(n)}{(1 - \beta_0 x(n) - \beta_1 x(n-1) - \alpha x(n))e^{-r(1 - \beta_0 x(n) - \beta_1 x(n-1))\frac{h^q}{q}} + \alpha x(n)}. \quad (10)$$

If we change of variables to $u_1(n) = x(n)$ and $u_2(n) = x(n-1)$, we get the two-dimensional discrete system as follows:

$$\begin{cases} u_1(n+1) = \frac{(1 - \beta_0 u_1(n) - \beta_1 u_2(n))u_1(n)}{(1 - \alpha u_1(n) - \beta_0 u_1(n) - \beta_1 u_2(n))e^{-r(1 - \beta_0 u_1(n) - \beta_1 u_2(n))\frac{h^q}{q}} + \alpha u_1(n)} \\ u_2(n+1) = u_1(n). \end{cases} \quad (11)$$

3. Stability analysis

Direct calculations show that the system (11) has a positive equilibrium point $E^* = (u_1^*, u_2^*) = (\frac{1}{\alpha + \beta_0 + \beta_1}, \frac{1}{\alpha + \beta_0 + \beta_1})$ and corresponding Jacobian matrix at this equilibrium point is

$$J(E^*) = \begin{pmatrix} \frac{e^{-\frac{\alpha r h^q}{(\alpha + \beta_0 + \beta_1)q}} (\alpha + \beta_0) - \beta_0}{\alpha} & \frac{\beta_1}{\alpha} (-1 + e^{-\frac{\alpha r h^q}{(\alpha + \beta_0 + \beta_1)q}}) \\ 1 & 0 \end{pmatrix}.$$

Moreover, the characteristic polynomial of $J(E^*)$ is given by:

$$p(\lambda) = \lambda^2 + \left(\frac{\beta_0}{\alpha} - e^{-\frac{\alpha r h^q}{(\alpha + \beta_0 + \beta_1)q}} (1 + \frac{\beta_0}{\alpha}) \right) \lambda - \frac{\beta_1}{\alpha} (e^{-\frac{\alpha r h^q}{(\alpha + \beta_0 + \beta_1)q}} - 1). \quad (12)$$

The following theorem confirms the stability of positive equilibrium point of the model (11) under some conditions.

Theorem 1. Let $\beta_0 > \alpha + \beta_1 > 2\alpha$. The following statements are true.

(i) Suppose that $3\beta_1 < \alpha + \beta_0$. The positive equilibrium point of system (11) is local asymptotically stable if and only if

$$0 < r < \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\alpha + \beta_0 - \beta_1}{\beta_0 - \alpha - \beta_1}\right). \quad (13)$$

(ii) Suppose that $3\beta_1 > \alpha + \beta_0$. The positive equilibrium point of system (11) is local asymptotically stable if and only if

$$0 < r < \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\beta_1}{\beta_1 - \alpha}\right). \quad (14)$$

Proof. By the Linearized Stability Theorem in [36] we get that the positive equilibrium point of the system (11) is locally asymptotically stable if and only if

$$\left| \frac{\beta_0}{\alpha} - e^{-\frac{\alpha r h^q}{(\alpha + \beta_0 + \beta_1)q}} (1 + \frac{\beta_0}{\alpha}) \right| < 1 - \frac{\beta_1}{\alpha} \left(e^{-\frac{\alpha r h^q}{(\alpha + \beta_0 + \beta_1)q}} - 1 \right) < 2. \quad (15)$$

The Eq. (15) can be considered in two cases as follows.

$$(a) \left| \frac{\beta_0}{\alpha} - e^{-\frac{\alpha h^q}{(\alpha+\beta_0+\beta_1)^q}} \left(1 + \frac{\beta_0}{\alpha}\right) \right| < 1 - \frac{\beta_1}{\alpha} \left(e^{-\frac{\alpha r}{\alpha+\beta_0+\beta_1}} - 1 \right)$$

$$(b) 1 - \frac{\beta_1}{\alpha} \left(e^{-\frac{\alpha h^q}{(\alpha+\beta_0+\beta_1)^q}} - 1 \right) < 2.$$

From (a), we will get

$$r < \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\alpha + \beta_0 - \beta_1}{\beta_0 - \alpha - \beta_1}\right) \quad (16)$$

where $\beta_0 > \beta_1 + \alpha$. Furthermore, from (b), we obtain

$$r < \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\beta_1}{\beta_1 - \alpha}\right) \quad (17)$$

where $\beta_1 > \alpha$. Considering (16) and (17) with the fact that for $3\beta_1 < \alpha + \beta_0$ we hold

$$0 < r < \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\alpha + \beta_0 - \beta_1}{\beta_0 - \alpha - \beta_1}\right) < \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\beta_1}{\beta_1 - \alpha}\right)$$

and for $3\beta_1 > \alpha + \beta_0$ one can obtain

$$0 < r < \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\beta_1}{\beta_1 - \alpha}\right) < \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\alpha + \beta_0 - \beta_1}{\beta_0 - \alpha - \beta_1}\right)$$

This completes our proof.

4. Bifurcation analysis

Bifurcation analysis is a very important issue for a better understanding mechanism of the biological models in both discrete and continuous time dynamical systems. In this section, we choose the parameter r as a bifurcation parameter for analyzing the Flip and Neimark–Sacker bifurcation of the system (11) at the equilibrium point E^* by using the center manifold and bifurcation theory in [28–31,37,38]

4.1. Flip bifurcation analysis

In this section, we discuss the existence and direction of flip bifurcation for the system.

Theorem 2. Suppose that the parameters satisfy $\beta_0 > \alpha + \beta_1 > 2\alpha$ and $3\beta_1 < \alpha + \beta_0$. If $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $r_1 = r_1^* = \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\alpha + \beta_0 - \beta_1}{\beta_0 - \alpha - \beta_1}\right)$ then the system (11) undergoes Flip bifurcation at the equilibrium point (u_1^*, u_2^*) . Moreover, if $\alpha_2 > 0$ then the period-2 solution is stable, and if $\alpha_2 < 0$ then the period-2 solution is unstable.

Proof. If $r_1^* = \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\alpha + \beta_0 - \beta_1}{\beta_0 - \alpha - \beta_1}\right)$, then the eigenvalues of Jacobian matrix at equilibrium point (u_1^*, u_2^*) are $\lambda_1 = -1$ and $\lambda_2 = \frac{2\beta_1}{\beta_1 - \alpha - \beta_0}$. The condition $|\lambda_1| \neq 1$ leads to $\alpha + \beta_0 \neq 3\beta_1$.

To decide the stability of the bifurcated period-2 points, we apply the center manifold reduction. Taking \bar{r} as an independent variable into the system (11) and making transformation: $u = u_1 - u_1^*$, $v = u_2 - u_2^*$ ve $\bar{r} = r - r_1^*$, then the system (11) is transformed into:

$$\begin{pmatrix} u \\ \bar{r} \\ v \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\alpha + \beta_0 + \beta_1}{\beta_1 - \beta_0 - \alpha} & 0 & \frac{2\beta_1}{\beta_1 - \beta_0 - \alpha} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ \bar{r} \\ v \end{pmatrix} + \begin{pmatrix} f(u, \bar{r}, v) \\ 0 \\ 0 \end{pmatrix} \quad (18)$$

where

$$f_1(u, \bar{r}, v) = m_{13}u^2 + m_{14}\bar{r}^2 + m_{15}v^2 + m_{16}u\bar{r} + m_{17}uv + m_{18}\bar{r}v + o((|u| + |\bar{r}| + |v|)^3)$$

and the Taylor coefficients m_{ij} are

$$m_{13} = \frac{(\alpha + \beta_0)(\alpha - \beta_0 + \beta_1)(\alpha + \beta_0 + \beta_1)(\alpha + \beta_0)(2\alpha - \ln\left(\frac{\beta_1 - \beta_0 - \alpha}{\alpha - \beta_0 + \beta_1}\right)\beta_0) + \ln\left(\frac{\beta_1 - \beta_0 - \alpha}{\alpha - \beta_0 + \beta_1}\right)\beta_0\beta_1}{\alpha^2(\alpha + \beta_0 - \beta_1)^2},$$

$$m_{14} = 0,$$

$$m_{15} = \frac{(\ln\left(\frac{\beta_1 - \beta_0 - \alpha}{\alpha - \beta_0 + \beta_1}\right) + \frac{2\alpha}{\beta_1 - \beta_0 - \alpha})(\alpha - \beta_0 + \beta_1)(\alpha + \beta_0 + \beta_1)\beta_1^2}{\alpha^2(\beta_1 - \beta_0 - \alpha)},$$

$$\begin{aligned}
 m_{16} &= \frac{h^q(\alpha + \beta_0)(\alpha - \beta_0 + \beta_1)}{q(\alpha^2 + 2\alpha\beta_0 + \beta_0^2 - \beta_1^2)}, \\
 m_{17} &= \frac{\beta_1(\alpha - \beta_0 + \beta_1)(\alpha + \beta_0 + \beta_1)}{\alpha^2(\alpha + \beta_0 - \beta_1)^2} \\
 &\quad \times ((\alpha + \beta_0)(\alpha(4 - \ln(\frac{\beta_1 - \beta_0 - \alpha}{\alpha - \beta_0 + \beta_1}))) + 2\ln(\frac{\beta_1 - \beta_0 - \alpha}{\alpha - \beta_0 + \beta_1})\beta_0) + \ln(\frac{\beta_1 - \beta_0 - \alpha}{\alpha - \beta_0 + \beta_1})(\alpha + 2\beta_0)\beta_1) \\
 m_{18} &= \frac{h^q\beta_1(\alpha - \beta_0 + \beta_1)}{q((\alpha + \beta_0)^2 - \beta_1^2)}.
 \end{aligned}$$

Let

$$T = \begin{pmatrix} 1 & 0 & \frac{2\beta_1}{\beta_1 - \beta_0 - \alpha} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and use the translation $\begin{pmatrix} u \\ \bar{r} \\ v \end{pmatrix} = T \begin{pmatrix} X \\ \mu \\ Y \end{pmatrix}$. Then the map (18) becomes

$$\begin{pmatrix} X \\ \mu \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2\beta_1}{\beta_1 - \beta_0 - \alpha} \end{pmatrix} \begin{pmatrix} X \\ \mu \\ Y \end{pmatrix} + \begin{pmatrix} F_1(X, \mu, Y) \\ 0 \\ F_2(X, \mu, Y) \end{pmatrix}, \tag{19}$$

where

$$\begin{aligned}
 F_1(X, \mu, Y) &= -\frac{(m_{13} + m_{16})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1}\mu^2 \\
 &\quad - \frac{(m_{13} + m_{15} - m_{17})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1}X^2 \\
 &\quad + \frac{(2m_{13} + m_{16} - m_{17} - m_{18})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1}X\mu \\
 &\quad - \frac{(m_{17} + m_{18})(\alpha + \beta_0 - \beta_1) - 2(2m_{13} + m_{16})\beta_1}{(\alpha + \beta_0 - 3\beta_1)(\alpha + \beta_0 - \beta_1)}Y\mu \\
 &\quad - \frac{m_{15}(\alpha + \beta_0 - \beta_1)^2 + 2\beta_1(-m_{17}(\alpha + \beta_0 - \beta_1)) + 2m_{13}\beta_1}{(\alpha + \beta_0 - 3\beta_1)(\alpha + \beta_0 - \beta_1)^2}Y^2 \\
 &\quad - \frac{2m_{15}(\alpha + \beta_0 - \beta_1) + 4m_{13}\beta_1 - m_{17}(\alpha + \beta_0 + \beta_1)}{\alpha + \beta_0 - 3\beta_1}XY
 \end{aligned}$$

and

$$\begin{aligned}
 F_2(X, \mu, Y) &= \frac{(m_{13} + m_{16})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1}\mu^2 \\
 &\quad + \frac{(m_{13} + m_{15} - m_{17})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1}X^2 \\
 &\quad - \frac{(2m_{13} + m_{16} - m_{17} - m_{18})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1}X\mu \\
 &\quad + \frac{(m_{17} + m_{18})(\alpha + \beta_0 - \beta_1) - 2(m_{13} + m_{16})\beta_1}{\alpha + \beta_0 - 3\beta_1}Y\mu \\
 &\quad + \frac{2m_{15}(\alpha + \beta_0 - \beta_1) + 4m_{13}\beta_1 - m_{17}(\alpha + \beta_0 + \beta_1)}{\alpha + \beta_0 - 3\beta_1}XY \\
 &\quad + \frac{m_{15}(\alpha + \beta_0 - \beta_1)^2 - 2\beta_1(m_{17}(\alpha + \beta_0 - \beta_1) - 2m_{13}\beta_1)}{(\alpha + \beta_0 - 3\beta_1)(\alpha + \beta_0 - \beta_1)}Y^2.
 \end{aligned}$$

Suppose that

$$W^c(0) = \{(X, \mu, Y) \in \mathbb{R}^3 \mid Y = h^*(X, \mu), h^*(0, 0) = 0, Dh^*(0, 0) = 0\}$$

is the center manifold for the system of $(X, Y) = (0, 0)$ near $\mu = 0$.

Assume that

$$h^*(X, \mu) = AX^2 + BX\mu + G\mu^2 + O((|X| + |\mu|)^3).$$

By approximate computation for the center manifold, we have

$$A = \frac{(m_{13} + m_{15} - m_{17})(\alpha + \beta_0 - \beta_1)^2}{(\alpha + \beta_0 - 3\beta_1)(\alpha + \beta_0 + \beta_1)}$$

$$B = \frac{(2m_{13} + m_{16} - m_{17} - m_{18})(\alpha + \beta_0 - \beta_1)^2}{(\alpha + \beta_0 - 3\beta_1)^2}$$

$$G = \frac{(m_{13} + m_{16})(\alpha + \beta_0 - \beta_1)^2}{(\alpha + \beta_0 - 3\beta_1)(\alpha + \beta_0 + \beta_1)}$$

Now, the map (19) restricted to the center manifold is given by

$$\tilde{F} : X \rightarrow -X + h_1X^2 + h_2X\mu + h_3\mu^2 + h_4X^3 + h_5X^2\mu + h_6X\mu^2 + h_7\mu^3 + O(|X| + |\mu|)^4 \quad (20)$$

where

$$h_1 = -\frac{(m_{13} + m_{15} - m_{17})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1}$$

$$h_2 = \frac{(2m_{13} + m_{16} - m_{17} - m_{18})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1}$$

$$h_3 = -\frac{(m_{13} + m_{16})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1}$$

$$h_4 = -\frac{(m_{13} + m_{15} - m_{17})(\alpha + \beta_0 - \beta_1)^2(2m_{15}(\alpha + \beta_0 - \beta_1) + 4m_{13}\beta_1 - m_{17}(\alpha + \beta_0 + \beta_1))}{(\alpha + \beta_0 - 3\beta_1)^2(\alpha + \beta_0 + \beta_1)}$$

$$h_5 = \frac{(\alpha + \beta_0 - \beta_1)^2}{(\alpha + \beta_0 - 3\beta_1)^3(\alpha + \beta_0 + \beta_1)} [-(-m_{16}m_{17} + m_{15}(2m_{16} - m_{17} - m_{18}) + m_{13}(4m_{15} - m_{17} + m_{18}))(\alpha + \beta_0)^2 - 2(2m_{13}^2 + 3m_{17}(m_{17} + m_{18}) + m_{13}(-2m_{15} + m_{16} - 4(m_{17} + m_{18}))) - m_{15}(m_{16} + 2(m_{17} + m_{18}))(\alpha + \beta_0)\beta_1 + (-20m_{13}^2 + m_{13}(-8m_{15} - 10m_{16} + 15m_{17} + m_{18}) + m_{17}(7m_{16} + 2(m_{17} + m_{18})) - m_{15}(4m_{16} + 5(m_{17} + m_{18})))\beta_1^2]$$

$$h_6 = -\frac{(\alpha + \beta_0 - \beta_1)^2}{(\alpha + \beta_0 - 3\beta_1)^3(\alpha + \beta_0 + \beta_1)} [(2m_{15}m_{16} + m_{16}m_{18} - (m_{17} + m_{18})^2)(\alpha + \beta_0)^2 - 2m_{16}(4m_{15} + m_{16} - 2m_{17} - m_{18})(\alpha + \beta_0)\beta_1 + (6m_{15}m_{16} - 2m_{16}^2 + (m_{17} + m_{18})^2 + m_{16}(4m_{17} + m_{18}))\beta_1^2 - 4m_{13}^2\beta_1(\alpha + \beta_0 + 5\beta_1) + m_{13}((2m_{15} + m_{17} + 2m_{18})(\alpha + \beta_0)^2 - 2(4m_{15} + 2m_{16} - 3m_{17} - 2m_{18})(\alpha + \beta_0)\beta_1 + (6m_{15} - 20m_{16} + 5m_{17} + 2m_{18})\beta_1^2)]$$

$$h_7 = \frac{(m_{13} + m_{16})(\alpha + \beta_0 - \beta_1)^2(-m_{17} + m_{18})(\alpha + \beta_0 - \beta_1) + 2(2m_{13} + m_{16})\beta_1}{(\alpha + \beta_0 - 3\beta_1)^2(\alpha + \beta_0 + \beta_1)}$$

As given by the flip bifurcation theorem in [39], the emergence of flip bifurcation for map (20) requires

$$\alpha_1 = \left[\frac{\partial F}{\partial \mu} \cdot \frac{\partial^2 F}{\partial X^2} + 2 \frac{\partial^2 F}{\partial X \partial \mu} \right] \Bigg|_{(0,0)} = \frac{2(2m_{13} + m_{16} - m_{17} - m_{18})(\alpha + \beta_0 - \beta_1)}{\alpha + \beta_0 - 3\beta_1} \neq 0$$

and

$$\alpha_2 = \left[\frac{1}{2} \cdot \left(\frac{\partial^2 F}{\partial X^2} \right)^2 + \frac{1}{3} \cdot \frac{\partial^3 F}{\partial X^3} \right] \Bigg|_{(0,0)} = \frac{2(m_{13} - m_{15})(m_{13} + m_{15} - m_{17})(\alpha + \beta_0 - \beta_1)^2}{(\alpha + \beta_0 - 3\beta_1)(\alpha + \beta_0 + \beta_1)} \neq 0.$$

Thus, the proof is completed.

4.2. Neimark–Sacker bifurcation analysis

As stated by the Neimark–Sacker bifurcation theorem in [39], the existence of the Neimark–Sacker bifurcation for the system requires some conditions that are eigenvalue assignment, transversality and nonresonance conditions. The following theorem is obtained for analyzing these conditions and it also give the direction of Neimark–Sacker bifurcation.

Theorem 3. Assume that the parameters satisfy $\beta_0 > \alpha + \beta_1 > 2\alpha$ and $3\beta_1 > \alpha + \beta_0$. If $\beta_1 \neq \alpha + \beta_0$, $2\beta_1 \neq \alpha + \beta_0$, $k \neq 0$ and $r = r_2^* = \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln\left(\frac{\beta_1}{\beta_1 - \alpha}\right)$ then the system (11) undergoes Neimark–Sacker bifurcation at the equilibrium point (u_1^*, u_2^*) . In addition to, if $k < 0$ then an attracting invariant cycle will appear for $r > r_2^*$, if $k > 0$ then a repelling invariant cycle will appear for $0 < r < r_2^*$.

Proof. Suppose that $r_2^* = \frac{q(\alpha + \beta_0 + \beta_1)}{\alpha h^q} \ln(\frac{\beta_1}{\beta_1 - \alpha})$. The characteristic equation of the model (11) at the equilibrium point $(u_1(r_2^*), u_2(r_2^*))$ is obtained by

$$\lambda^2 + \frac{\alpha + \beta_0 - \beta_1}{\beta_1} \lambda + 1 = 0. \tag{21}$$

The eigenvalues of Eq. (21) are

$$\lambda_{1,2}(r_2^*) = \frac{\beta_1 - \beta_0 - \alpha}{2\beta_1} \pm i \frac{\sqrt{(\alpha + \beta_0 - 3\beta_1)(\alpha + \beta_0 + \beta_1)}}{2\beta_1} = a \pm ib.$$

We get $|\lambda_{1,2}(r_2^*)| = 1$ and the transversality condition gives to the inequality

$$\left. \frac{d|\lambda_{1,2}(r)|}{dr} \right|_{r=r_2^*} = - \frac{h^q(\alpha - \beta_1)(\alpha^2 + \beta_0^2 + \beta_0(2\alpha - \beta_1) - \alpha\beta_1 - \beta_1^2)}{q\beta_1(\alpha + \beta_0 + \beta_1)\sqrt{2(\alpha^2 + \beta_0^2 + 2\beta_0(\alpha - \beta_1) - 2\alpha\beta_1 - \beta_1^2)}} \neq 0.$$

Moreover, if $\beta_1 \neq \alpha + \beta_0$ and $2\beta_1 \neq \alpha + \beta_0$ then we have $\lambda_{1,2}^n(r_2^*) \neq 1$ ($n = 1, 2, 3, 4$).

Let $u = u_1 - u_1^*$ and $v = u_2 - u_2^*$, then the system (11) is transformed into

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\beta_1 - \beta_0 - \alpha}{\beta_1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u, v) \\ 0 \end{pmatrix} \tag{22}$$

where

$$f(u, v) = s_{13}u^2 + s_{14}uv + s_{15}v^2 + s_{16}u^3 + s_{17}u^2v + s_{18}uv^2 + s_{19}v^3 + o((|u| + |v|)^4) \tag{23}$$

and the Taylor coefficients s_{ij} are

$$\begin{aligned} s_{13} &= \frac{(\alpha + \beta_0)(\alpha - \beta_1)(\alpha + \beta_0 + \beta_1)(\alpha^2 + \beta_0(\alpha - \ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_1))}{\alpha^2\beta_1^2}, \\ s_{14} &= \frac{(\alpha - \beta_1)(\alpha + \beta_0 + \beta_1)(2\alpha(\alpha + \beta_0) - \ln(\frac{\beta_1}{\beta_1 - \alpha})(\alpha + 2\beta_0)\beta_1)}{\alpha^2\beta_1}, \\ s_{15} &= \frac{(\alpha - \beta_1)(\alpha + \beta_0 + \beta_1)(\alpha - \ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_1)}{\alpha^2}, \\ s_{16} &= \frac{(\alpha + \beta_0)(\beta_1 - \alpha)(\alpha + \beta_0 + \beta_1)^2}{2\alpha^3\beta_1^3} \\ &\quad \times \left[2\alpha^2(\alpha + \beta_0)^2 - 2\alpha(1 + 2\ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_0(\alpha + \beta_0)\beta_1) + \ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_0(2\alpha + (2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0)\beta_1^2 \right], \\ s_{17} &= \frac{(\beta_1 - \alpha)(\alpha + \beta_0 + \beta_1)^2}{2\alpha^3\beta_1^2} \\ &\quad \times [2\alpha(\alpha + \beta_0)(3\beta_0(\alpha - \beta_1) + \alpha(3\alpha - \beta_1)) - 2\ln(\frac{\beta_1}{\beta_1 - \alpha})(\alpha + \beta_0)(\alpha + 3\beta_0)(2\alpha - \beta_1)\beta_1 \\ &\quad + \ln(\frac{\beta_1}{\beta_1 - \alpha})^2\beta_0(2\alpha + 3\beta_0)\beta_1^2], \\ s_{18} &= \frac{(\beta_1 - \alpha)(\alpha + \beta_0 + \beta_1)^2}{2\alpha^3\beta_1} \\ &\quad \times \left[6\alpha^2(\alpha + \beta_0) - 2\alpha(2\alpha + 3\beta_0)\beta_1 - 2\ln(\frac{\beta_1}{\beta_1 - \alpha})(2\alpha + 3\beta_0)(2\alpha - \beta_1)\beta_1 + \ln(\frac{\beta_1}{\beta_1 - \alpha})^2(\alpha + 3\beta_0)\beta_1^2 \right], \\ s_{19} &= \frac{(\beta_1 - \alpha)(\alpha + \beta_0 + \beta_1)^2(2\alpha^2 - 2(\alpha + 2\alpha\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_1 + \ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_1^2)}{2\alpha^3}. \end{aligned}$$

Let

$$T = \begin{pmatrix} \frac{\sqrt{(3\beta_1 - \beta_0 - \alpha)(\alpha + \beta_0 + \beta_1)}}{2\beta_1} & \frac{\beta_1 - \beta_0 - \alpha}{2\beta_1} \\ 0 & 1 \end{pmatrix}$$

and use the translation $\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} X \\ Y \end{pmatrix}$ Then the map (22) becomes

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} F(X, Y) \\ 0 \end{pmatrix} \quad (24)$$

where

$$F(X, Y) = g_{13}X^2 + g_{14}XY + g_{15}Y^2 + g_{16}X^3 + g_{17}X^2Y + g_{18}XY^2 + g_{19}Y^3 + o(|X| + |Y|)^4,$$

and the Taylor coefficients g_{ij} are

$$\begin{aligned} g_{13} &= \frac{(\alpha + \beta_0)(\alpha - \beta_1)(\alpha + \beta_0 + \beta_1)\sqrt{(\alpha + \beta_0 + \beta_1)(3\beta_1 - \beta_0 - \alpha)}(\alpha^2 + \beta_0(\alpha - \ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_1))}{2\alpha^2\beta_1^3}, \\ g_{14} &= \frac{1}{\alpha^2\beta_1^3}[-\alpha^2(\alpha + \beta_0)^4 + \alpha(\alpha + \beta_0)^3(\alpha + (1 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0)\beta_1 + (\alpha + \beta_0)^2(3\alpha^2 - \ln(\frac{\beta_1}{\beta_1 - \alpha})) \\ &\quad \times \beta_0(\alpha + \beta_0)\beta_1^2 - \alpha(1 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))(\alpha + \beta_0)(\alpha + 3\beta_0)\beta_1^3 + (-2\alpha^2 + 2\alpha(-1 + \ln(\frac{\beta_1}{\beta_1 - \alpha})))\beta_0 \\ &\quad + 3\ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_0^2\beta_1^4 + \ln(\frac{\beta_1}{\beta_1 - \alpha})(\alpha + 2\beta_0)\beta_1^5], \\ g_{15} &= \frac{(\beta_1 - \alpha)(\alpha + \beta_0 - 2\beta_1)(\alpha + \beta_0 + \beta_1)^2}{2\alpha^2\beta_1^3\sqrt{(\alpha + \beta_0 + \beta_1)(3\beta_1 - \beta_0 - \alpha)}}[\alpha(\alpha + \beta_0)^2 - (\alpha + \beta_0)(\alpha + \ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_0)\beta_1 \\ &\quad + (-2\alpha + \ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_0)\beta_1^2 + \ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_1^3], \\ g_{16} &= \frac{(\alpha + \beta_0)(\alpha + \beta_0 - 3\beta_1)(\alpha - \beta_1)(\alpha + \beta_0 + \beta_1)^3}{8\alpha^3\beta_1^5} \\ &\quad \times [2\alpha^2(\alpha + \beta_0)^2 - 2\alpha(1 + 2\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0(\alpha + \beta_0)\beta_1 + \ln(\frac{\beta_1}{\beta_1 - \alpha})\beta_0(2\alpha + (2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0)\beta_1^2], \\ g_{17} &= \frac{(\alpha - \beta_1)(\alpha + \beta_0 + \beta_1)^2\sqrt{(\alpha + \beta_0 + \beta_1)(3\beta_1 - \beta_0 - \alpha)}}{8\alpha^3\beta_1^5} \\ &\quad \times [6\alpha^2(\alpha + \beta_0)^4 - 6\alpha(\alpha + \beta_0)^3(\alpha + (1 + 2\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0)\beta_1 - 3(\alpha + \beta_0)^2(4\alpha^2 - 2(\alpha + 3\alpha\ln(\frac{\beta_1}{\beta_1 - \alpha}))) \\ &\quad \times \beta_0 - \ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2\beta_1^2 + (\alpha + \beta_0)(4\alpha^2(1 + 2\ln(\frac{\beta_1}{\beta_1 - \alpha})) + 6\alpha(2 + 3\ln(\frac{\beta_1}{\beta_1 - \alpha})))\beta_0 \\ &\quad - 3\ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2\beta_1^3 - 2\ln(\frac{\beta_1}{\beta_1 - \alpha})(2\alpha^2 + 2\alpha(4 + \ln(\frac{\beta_1}{\beta_1 - \alpha})))\beta_0 \\ &\quad + 3(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2\beta_1^4], \\ g_{18} &= \frac{(\beta_1 - \alpha)(\alpha + \beta_0 + \beta_1)^2}{8\alpha^3\beta_1^5}[6\alpha^2(\alpha + \beta_0)^5 - 6\alpha(\alpha + \beta_0)^4(2\alpha + (1 + 2\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0)\beta_1 - 3(\alpha + \beta_0)^3 \\ &\quad \times (6\alpha^2 - 2\alpha(2 + 5\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0 - \ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2)\beta_1^2 + 2(\alpha + \beta_0)^2(8\alpha^2(2 \\ &\quad + \ln(\frac{\beta_1}{\beta_1 - \alpha})) + 3\alpha(3 + 4\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0 - 3\ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2)\beta_1^3 - (\alpha + \beta_0) \\ &\quad \times (8\alpha^2(-2 + 3\ln(\frac{\beta_1}{\beta_1 - \alpha})) + 2\alpha(12 + \ln(\frac{\beta_1}{\beta_1 - \alpha})(37 + 4\ln(\frac{\beta_1}{\beta_1 - \alpha})))\beta_0 + 9\ln(\frac{\beta_1}{\beta_1 - \alpha})) \\ &\quad \times (2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2\beta_1^4 - 4(2\alpha^2(2 + 3\ln(\frac{\beta_1}{\beta_1 - \alpha})) - 2\alpha(-3 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))(1 + \ln(\frac{\beta_1}{\beta_1 - \alpha})))\beta_0 \\ &\quad - 3\ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2\beta_1^5 + 4\ln(\frac{\beta_1}{\beta_1 - \alpha})(\alpha(4 + \ln(\frac{\beta_1}{\beta_1 - \alpha})) + 3(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0)\beta_1^5], \\ g_{19} &= \frac{(\alpha + \beta_0 - 2\beta_1)(\alpha - \beta_1)(\alpha + \beta_0 + \beta_1)^3}{8\alpha^3\beta_1^5\sqrt{(\alpha + \beta_0 + \beta_1)(3\beta_1 - \beta_0 - \alpha)}} \\ &\quad \times [2\alpha^2(\alpha + \beta_0)^4 - 2\alpha(\alpha + \beta_0)^3(2\alpha + (1 + 2\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0)\beta_1 - (\alpha + \beta_0)^2(6\alpha^2 - 2\alpha \end{aligned}$$

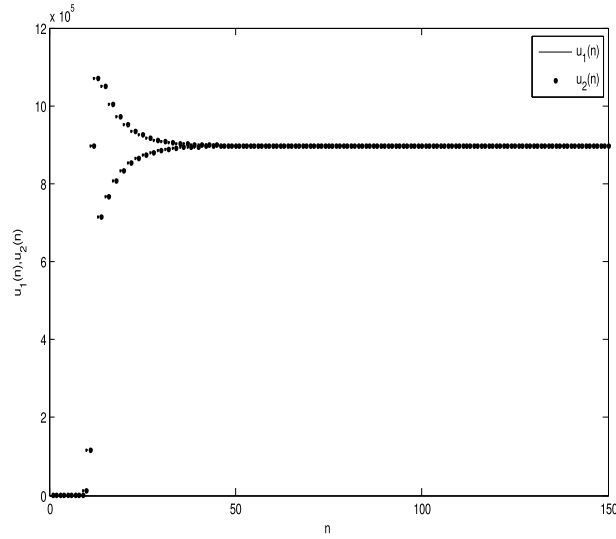


Fig. 1. Stable equilibrium point of the system (11) for parameter values $\alpha = 0.14 \times 10^{-7}$, $\beta_0 = 1.0005 \times 10^{-6}$, $\beta_1 = 0.9855 \times 10^{-7}$, $q = 0.5$, $h = 0.15$ and $r = 3$.

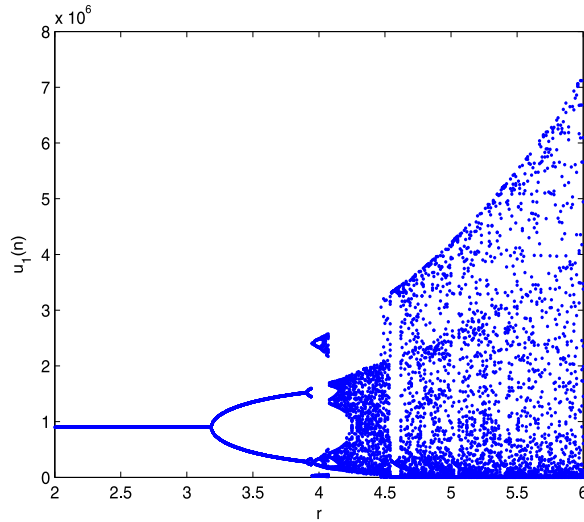


Fig. 2. Flip bifurcation diagram of the system (11) for the parameter values $\alpha = 0.14 \times 10^{-7}$, $\beta_0 = 1.0005 \times 10^{-6}$, $\beta_1 = 0.9855 \times 10^{-7}$, $q = 0.5$ and $h = 0.15$.

$$\begin{aligned} & \times (2 + 5\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0 - \ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2\beta_1^2 + 2(\alpha + \beta_0)(2\alpha^2(3 + 2\ln(\frac{\beta_1}{\beta_1 - \alpha}))) \\ & + \alpha(3 + 4\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0 - \ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2\beta_1^3 - (4\alpha^2(-1 + 3\ln(\frac{\beta_1}{\beta_1 - \alpha}))) \\ & + 2\alpha(4 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))(13 + 2\ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0 + 3\ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0^2\beta_1^4 \\ & + 4(-\alpha(2 + 3\ln(\frac{\beta_1}{\beta_1 - \alpha})) + \ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_0)\beta_1^5 + 4\ln(\frac{\beta_1}{\beta_1 - \alpha})(2 + \ln(\frac{\beta_1}{\beta_1 - \alpha}))\beta_1^6]. \end{aligned}$$

Now the coefficient k , which determines the direction of the invariant curve, can be computed

$$k = -\text{Re} \left[\frac{(1 - 2\lambda)\bar{\lambda}^2}{1 - \lambda} \xi_{11}\xi_{20} \right] - \frac{1}{2} (|\xi_{11}|^2 - |\xi_{02}|^2 + \text{Re}(\bar{\lambda}\xi_{21})) \tag{25}$$

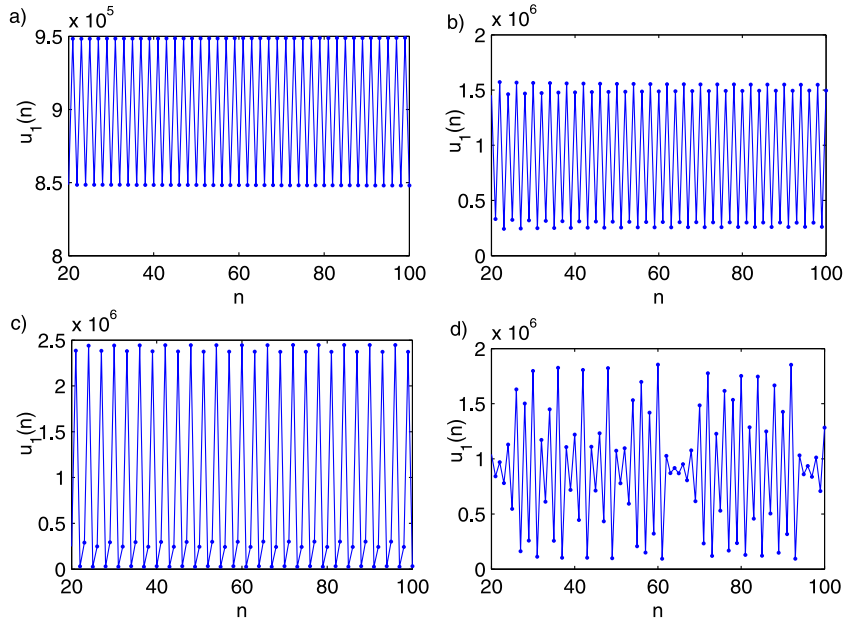


Fig. 3. Time series plot of model (11) with respect to Fig. 2: (a) period-two orbit for $r = 3.19$, (b) period-four orbit for $r = 3.91$, (c) period-eight orbit for $r = 3.97$, (d) chaotic for $r = 4.3$.

where

$$\begin{aligned} \xi_{20} &= \frac{1}{8} [(F_{1XX} - F_{1YY} + 2F_{2XY}) + i(F_{2XX} - F_{2YY} - 2F_{1XY})], \\ \xi_{11} &= \frac{1}{4} [(F_{1XX} + F_{1YY}) + i(F_{2XX} + F_{2YY})], \\ \xi_{02} &= \frac{1}{8} [(F_{1XX} - F_{1YY} - 2F_{2XY}) + i(F_{2XX} - F_{2YY} + 2F_{1XY})], \\ \xi_{21} &= \frac{1}{16} [(F_{1XXX} + F_{1XXY} + F_{2XXY} + F_{2YY}) + i(F_{2XXX} + F_{2XY} - F_{1XXY} - F_{1YY})]. \end{aligned}$$

Thus the proof of Theorem 3 is completed.

5. Numerical simulations

In this section, we present some numerical simulations to demonstrate the accuracy of the theoretical results obtained in Section 4. Therefore, the parameter r is chosen as a bifurcation parameter and the other parameters of model are fixed. The bifurcation parameters are considered as in the following cases:

Case(i) varying r in range $2 \leq r \leq 6$ and fixing $\alpha = 0.14 \times 10^{-7}$, $\beta_0 = 1.0005 \times 10^{-6}$, $\beta_1 = 0.9855 \times 10^{-7}$, $q = 0.5$ and $h = 0.15$. From Theorem 2, the critical flip bifurcation value is obtained as $r_1^* \approx 3.18656$. In this situation, $\alpha_1 = -1.57475 \neq 0$ and $\alpha_2 = 5.93602 \times 10^{-15} \neq 0$. Now we can say that Flip bifurcation appears around the fixed point $E^* = (8.98432 \times 10^{-6}, 8.98432 \times 10^{-6})$ for the model (11). Moreover, since $\alpha_2 > 0$ the period-2 orbits bifurcate from E^* are stable (Figs. 2 and 3a). We note that for the above parameter values the eigenvalues of the Jacobian matrix are $\lambda_1 = 1$ and $\lambda_2 = 0.215186$. In addition we calculate the maximum Lyapunov exponents corresponding Fig. 2 and plot in Fig. 4 where some Lyapunov exponents are bigger than 0, some are smaller than 0. Therefore there exist chaotic regions and period orbits in the parametric space.

Case(ii) varying r in range $1 \leq r \leq 4$ and fixing $\alpha = 0.14 \times 10^{-7}$, $\beta_0 = 1.0005 \times 10^{-6}$, $\beta_1 = 2.9855 \times 10^{-6}$, $q = 0.5$ and $h = 0.15$. From Theorem 3, the critical Neimark–Sacker bifurcation value can be calculated as $r_2^* \approx 1.73375$. For $r = r_2^*$, transversality and nonresonance condition leads to $\left. \frac{d|\lambda_{1,2}(r)|}{dr} \right|_{r=r_2^*} = 0.398326i \neq 0$ and $p(r_2^*) = -0.660191 \neq 0, 1$,

respectively. Now Neimark–Sacker bifurcation comes out from the fixed point $(2.5 \times 10^5, 2.5 \times 10^5)$ at $r_2^* = 1.73375$ for the model (11) (Fig. 6). In this situation, the norm of eigenvalues of the Jacobian matrix is $|\lambda_{1,2}| = |0.330095 \pm 0.943948i| = 1$ and the coefficients ξ_{ij} are $\xi_{20} = -5.31745 \times 10^{-7} + 8.44131 \times 10^{-7}i$, $\xi_{11} = 1.10931 \times 10^{-9}$, $\xi_{02} = -5.31745 \times 10^{-7} - 8.44130 \times 10^{-7}i$ and $\xi_{21} = 1.33632 \times 10^{-12} - 4.69503 \times 10^{-13}i$. From [39], the critical real part is obtained as

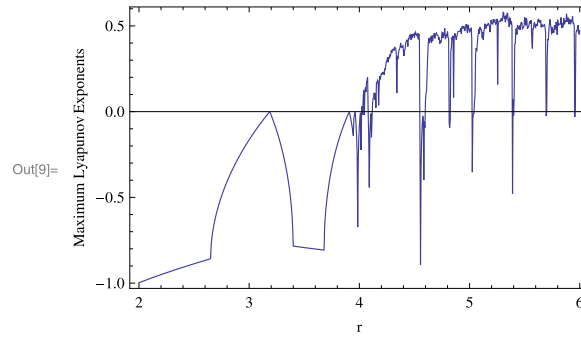


Fig. 4. Maximum Lyapunov exponents of the system (11) corresponding to Fig. 2.

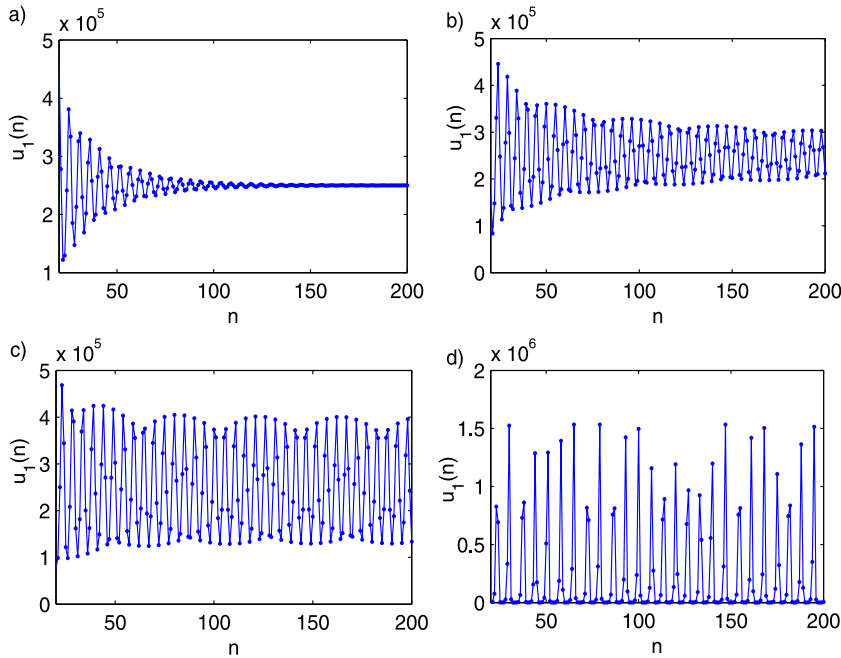


Fig. 5. Time series plot of model (11) with respect to Fig. 6: (a) asymptotically stable for $r = 1.6$, (b) quasi-periodic solution for $r = 1.73375$, (c) quasi-periodic solution for $r = 1.8$, (d) chaotic for $r = 3.4$.

$k = -9.98695 \times 10^{-13}$. Therefore, a supercritical Neimark–Sacker bifurcation occurs at r_2^* (Fig. 6). In order to see the chaotic dynamics, we calculate the maximum Lyapunov exponents corresponding Fig. 6 and plot in Fig. 7. The sign of the maximum Lyapunov exponent confirms the existence of the strange attractor.

6. Results and discussion

In this work, we have considered a conformable fractional order differential equations with piecewise constant arguments model (5) for modeling bacteria population model in a microcosm. We apply a discretization process to the model (5) and obtain two dimensional discrete dynamical system (11). Thus, the fractional order parameter q is included as a new parameter into the system of difference equations. By using the Schur Cohn criterion the necessary and sufficient stability conditions of the model according the parameter r is obtained and given in Theorem 1. The parameter values are taken in [4] in terms of consistency with the biological facts as $\alpha = 0.14 \times 10^{-7}$, $\beta_0 = 1.0005 \times 10^{-6}$, $\beta_1 = 0.9855 \times 10^{-6}$, $q = 0.5$ and $h = 0.15$ and initial condition for bacteria population is nearly 10^{-4} (cells/ml). Theorem 1 gives two stability regions according the bacteria population growth rate (r). The first stability region obtained from Theorem1(i) is $r < 3.18656 = r_1^*$. If bacteria population growth rate is chosen $r = 3$, then the bacteria population increase and eventually reaches to stable equilibrium point 8.98432×10^5 that is homogeneous bacteria distributions (Fig. 1). Let $\beta_1 = 2.9855 \times 10^{-6}$. In this situation stability region is obtained as $r < 1.73375 = r_2^*$ from Theorem 1(ii).

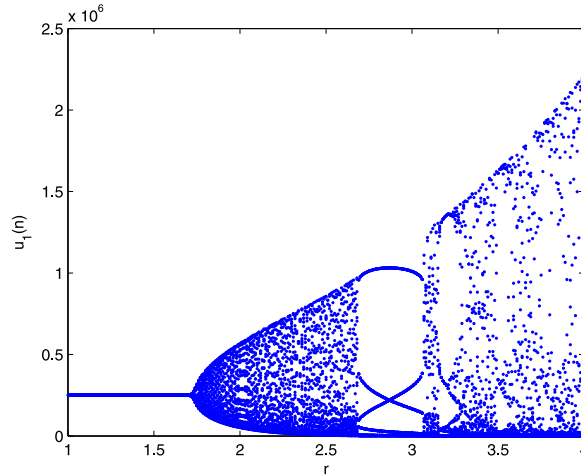


Fig. 6. Neimark-Sacker bifurcation diagram of the system (11) for $\alpha = 0.14 \times 10^{-7}$, $\beta_0 = 1.0005 \times 10^{-6}$, $\beta_1 = 2.9855 \times 10^{-6}$, $q = 0.5$ and $h = 0.15$.

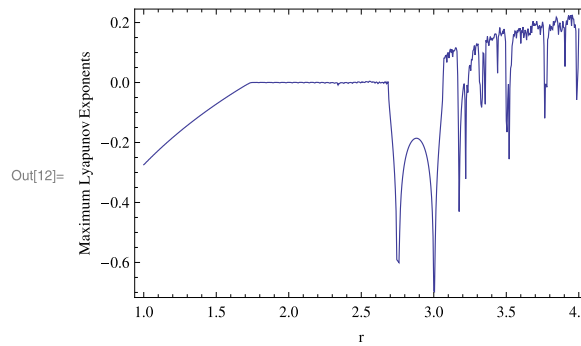


Fig. 7. Maximum Lyapunov exponents of the system (11) corresponding to Fig. 6.

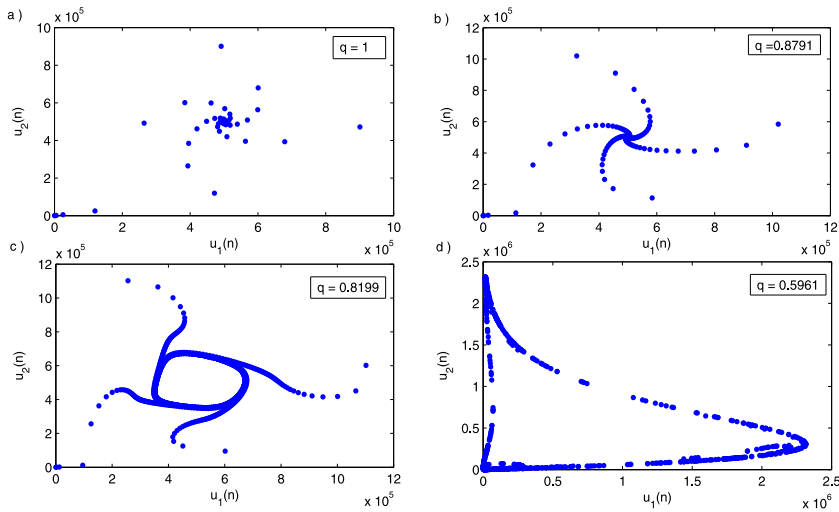


Fig. 8. Phase portraits for different values of q for the system (11).

By using the center manifold theorem and the bifurcation theory we show that the discrete system (11) undergoes both a Flip bifurcation and a Neimark-Sacker bifurcation around the positive equilibrium point. If bacteria population growth rate passes a threshold value $r_1^* = 3.18656$, the system undergoes a Flip bifurcation and after this chaotic dynamics

occurs for a bacteria population that is inhomogeneous spatial population distributions (Figs. 2, 3, 4). From these figures, we observe the existence many attractors (e.g., steady state, period-2 orbit, period-4 orbit, period-8 orbit, chaos etc.) with the variation parameter r . Therefore the model is sensitive to initial conditions and therefore model (11) is multistable system. In addition, for $r_2^* = 1.73375$ a stable limit cycle is formed around the equilibrium point as a result of Neimark–Sacker bifurcation (Figs. 5, 6, 7). From these figures, we also observe the existence quasi-periodic solution and chaos for the model (11) with the variation parameter r . This is another indication that the system is multistable. Multistable system is also seen in study [40,41].

On the other hand, the effect of the change of fractional order derivative parameter q on system (11) is illustrated in Fig. 8. This figure shows that the stable behavior of the system is destabilizing when decreasing the parameter q . Fig. 8 demonstrate phase portraits of the model and clearly depicts how a smooth invariant circle bifurcates from the stable equilibrium point ($2.5 \times 10^5, 2.5 \times 10^5$). The equilibrium point of the model is stable for $q < 0.8199$ (Fig. 8a–b), that is loses its stability $q = 0.8199$ through the Neimark–Sacker bifurcation (Fig. 8c), and that an invariant circle appears when the parameter q exceeds 0.8199. After this, its radius becomes larger with the decrease of parameter q and chaotic attractor is formed around the equilibrium point (Fig. 8d).

Our model (5) is a more general model than model (4) in that it includes local fractional order derivative q and discretization parameter h . We note that if we choose $q = 1$ and $h = 1$ in model (5), the model is reduced to model (4). However, since model (5) is a local fractional order, it includes local memory effects for a bacteria population that cannot be reflected by integer order model (4). Therefore, the model exhibits richer dynamic behaviors than the model (4) according to the different states of the local fractional order parameter q . In our model, Neimark–Sacker bifurcation occurs around the positive equilibrium point for $q = 0.8199$ (Fig. 8). It is not possible to observed this situation in model (4) because it does not include local fractional order parameter q . This is an expected result when we compare fractional order dynamical systems with integer order counterpart.

7. Conclusion

In this study, we consider a bacteria population model consisting of conformable fractional order differential equation with piecewise constant arguments. Since the model (5) is a local fractional order, it contains the local memory effect. This shows the advantage of our model compared to model (4) formed by ordinary differential equations. We also use piecewise constant arguments including control parameter h which represents the delay effect in the population. Thus, discretization parameter h take control the length of the subintervals $[nh, (n+1)h)$ that give us to controlling dynamical behavior of the discrete model (11). It is observed that the discrete model has quite rich dynamic behaviors such as period-2, period-4, period-8, quasi-periodic solutions and chaos according to the change of parameter r which demonstrates bacteria population growth rate. Periodic solutions lead to flip bifurcation and quasi periodic solutions cause to Neimark–Sacker bifurcation, that resulting in chaotic behavior. Due to the increase of the parameter r , quasi-periodic solutions first lead to damped oscillatory solutions, followed by stable limit cycles and then chaotic behaviors. These dynamic behaviors explain some biological phenomena such as homogeneous and inhomogeneous spatial population distributions for the bacteria population and provide a recipe for controlling the bacteria population. In addition, changing of fractional parameter q in the system show that local memory effect causes significant structural changes such as Neimark–Sacker bifurcation and chaos in dynamical behavior of the model.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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